

## The Gravity Field of the Earth - Part 1

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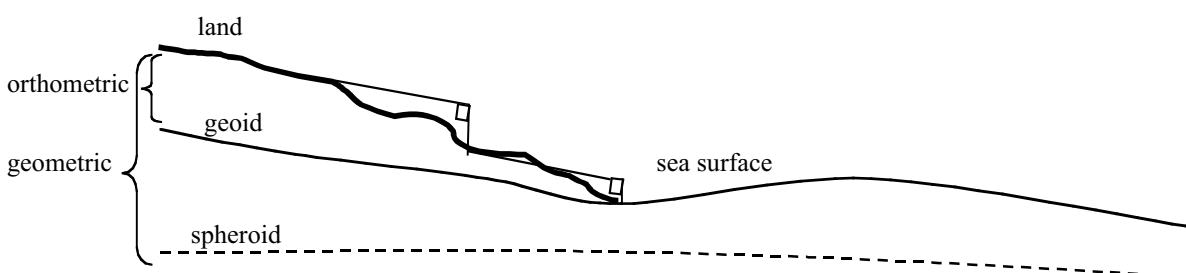
This chapter covers *physical geodesy* - the shape of the Earth and its gravity field. This is just electrostatic theory applied to the Earth. Unlike electrostatics, geodesy is a nightmare of unusual equations, unusual notation, and confusing conventions. There is no clear and concise book on the topic although Chapter 5 of Turcotte and Schubert is OK.

The things that make physical geodesy messy include:

- earth rotation
- latitude is measured from the equator instead of the pole;
- latitude is not the angle from the equator but is referred to the ellipsoid;
- elevation is measured from a theoretical surface called the geoid;
- spherical harmonics are defined differently from standard usage;
- anomalies are defined with respect to an ellipsoid having parameters that are constantly being updated;
- there are many types of anomalies related to various derivatives of the potential; and
- mks units are not commonly used in the literature.

In the next couple of lectures, I'll try to present this material with as much simplification as possible. Part of the reason for the mess is that prior to the launch of artificial satellites, measurements of elevation and gravitational acceleration were all done on the surface of the Earth (land or sea). Since the shape of the Earth is linked to variations in gravitational potential, measurements of acceleration were linked to position measurements both physically and in the mathematics. Satellite measurements are made in space well above the complications of the surface of the earth, so most of these problems disappear. Here are the two most important issues related to old-style geodesy.

### *Elevation*



Prior to satellites and the global positioning system (GPS), elevation was measured with respect to sea level - *orthometric height*. Indeed, elevation is still defined in this way however, most measurements are made with GPS. The pre-satellite approach to measuring elevation is called leveling.

1. Start at sea level and call this zero elevation. (If there were no winds, currents and tides then the ocean surface would be an equipotential surface and all shorelines would be at exactly the same potential.)
2. Sight a line inland perpendicular to a plumb line. Note that this plumb line will be perpendicular to the equipotential surface and thus is not pointed toward the geocenter.
3. Measure the height difference and then move the setup inland and to repeat the measurements until you reach the next shoreline. If all measurements are correct you will be back to zero elevation assuming the ocean surface is an equipotential surface.

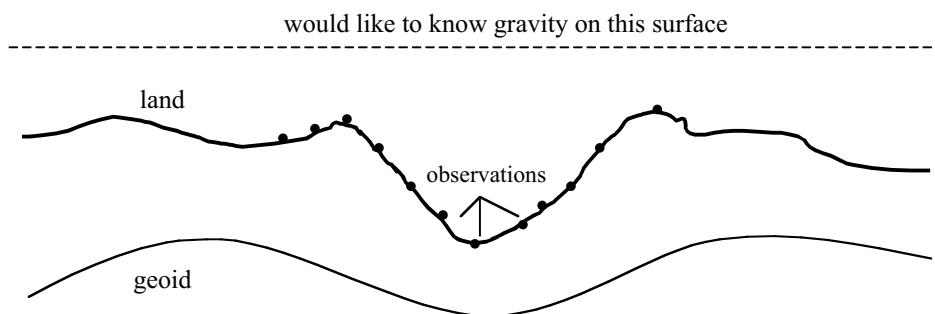
After artificial satellites measuring *geometric height* is easier, especially if one is far from a coastline.

1. Calculate the x,y,z position of each GPS satellite in the constellation using a global tracking network.
2. Measure the travel time to 4 or more satellites, 3 for position and one for clock error.
3. Establish your x,y,z position and convert this to height above the *spheroid* which we'll define below.
4. Go to a table of geoid height and subtract the local geoid height to get the orthometric height used by all surveyors and mappers.

Orthometric heights are useful because water flows downhill in this system while it does not always flow downhill in the geometric height system. Of course the problem with orthometric heights is that they are very difficult to measure or one must have a precise measurement of geoid height. Let *geodesists* worry about these issues.

### Gravity

The second complication in the pre-satellite geodesy is the measurement of gravity. Interpretation of surface gravity measurement is either difficult or trivial depending whether you are on land or at sea, respectively. Consider the land case illustrated below.



Small variations in the acceleration of gravity ( $< 10^{-6} g$ ) can be measured on the land surface. The major problem is that when the measurement is made in a valley, there are masses above the observation plane. Thus, bringing the gravity measurement to a common level requires

knowledge of the mass distribution above the observation point. This requires knowledge of both the geometric topography and the 3-D density. We could assume a constant density and use leveling to get the orthometric height but we need to convert to geometric height to do the gravity correction. To calculate the geometric height we need to know the geoid but the geoid height measurement comes from the gravity measurement so there is no exact solution. Of course one can make some approximations to get around this dilemma but it is still a problem and this is the fundamental reason why many geodesy books are so complicated. Of course if one could make measurements of both the gravity and topography on a plane (or sphere) above all of the topography, our troubles would be over.

Ocean gravity measurements are much less of a problem because the ocean surface is nearly equal to the geoid so we can simply define the ocean gravity measurement as *free-air gravity*. We'll get back to all of this again later when we discuss flat-earth approximations for gravity analysis.

**GLOBAL GRAVITY** (References: Turcotte and Schubert *Geodynamics*, Ch. 5. p. 198-212; Stacey, *Physics of the Earth*, Ch. 2+3; Jackson, *Electrodynamics*, Ch. 3; Fowler, *The Solid Earth*, Ch. 5.)

The gravity field of the Earth can be decomposed as follows:

- the main field due to the total mass of the earth;
- the second harmonic due to the flattening of the Earth by rotation; and
- anomalies which can be expanded in spherical harmonics or fourier series.

The combined main field and the second harmonic make up the reference earth model (i.e., spheroid, the reference potential, and the reference gravity). Deviations from this reference model are called elevation, geoid height, deflections of the vertical, and gravity anomalies.

### *Spherical Earth Model*

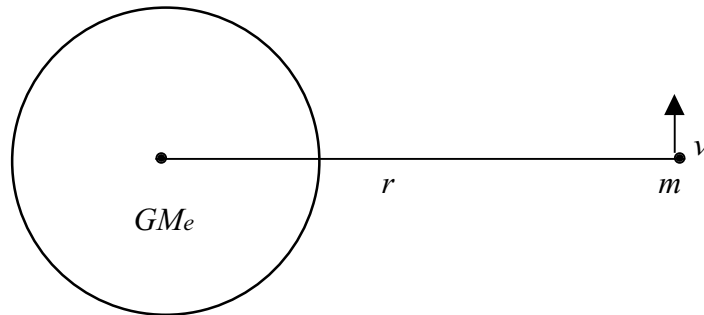
The spherical earth model is a good point to define some of the unusual geodetic terms. There are both fundamental constants and derived quantities.

parameter	description	formula	value/unit
$R_e$	mean radius of earth	-	6371000 m
$M_e$	mass of earth	-	$5.98 \times 10^{24}$ kg
$G$	gravitational constant	-	$6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
$\rho$	mean density of earth	$M_e (4/3\pi R_e^3)^{-1}$	$5520 \text{ kg m}^{-3}$
$U$	mean potential energy needed to take a unit mass from the surface of the earth and place it at infinite distance	$-GM_e R_e^{-1}$	$-6.26 \times 10^{-7} \text{ m}^2 \text{ s}^{-2}$
$g$	mean acceleration on the surface of the earth	$-\delta U / \delta r = -GM_e R_e^{-2}$	$-9.82 \text{ m s}^{-2}$
$\delta g / \delta r$	gravity gradient or free-air correction	$\delta g / \delta r = -2GM_e R_e^{-3}$ $= -2g / R_e$	$3.086 \times 10^{-6} \text{ s}^{-2}$

We should say a little more about units. Deviations in acceleration from the reference model, described next, are measured in units of milligal ( $1 \text{ mgal} = 10^{-3} \text{ cm s}^{-2} = 10^{-5} \text{ m s}^{-2} = 10 \text{ gravity units (gu)}$ ). As notes above the vertical gravity gradient is also called *the free-air correction* since it is the first term in the Taylor series expansion for gravity about the radius of the earth.

$$g(r) = g(R_e) + \frac{\partial g}{\partial r}(r - R_e) + \dots \quad (1)$$

*Example:* How does one measure the mass of the Earth? The best method is to time the orbital period of an artificial satellite. Indeed measurements of all long-wavelength gravitational deviations from the reference model are best done by satellites.



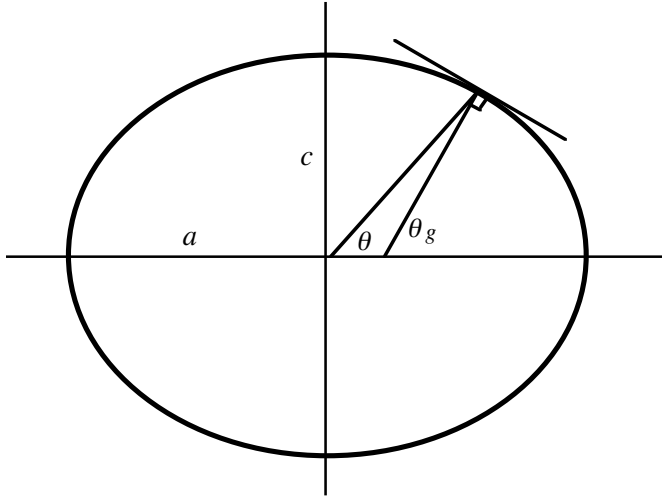
$$\frac{GM_e m}{r^2} = \frac{mv^2}{r} \quad (2)$$

$$\frac{GM_e}{r^2} = r\omega_s^2 \quad \Rightarrow \quad GM_e = r^3\omega_s^2$$

The mass is in orbit about the center of the Earth so the outward centrifugal force is balanced by the inward gravitational force; this is Kepler's third law. If we measure the radius of the satellite orbit  $r$  and the orbital frequency  $\omega_s$ , we can estimate  $GM_e$ . For example the satellite Geosat has a orbital radius of 7168 km and a period of 6037.55 sec so  $GM_e$  is  $3.988708 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$ . Note that the product  $GM_e$  is tightly constrained by the observations but that the accuracy of the mass of the earth  $M_e$  is related to the accuracy of the measurement of  $G$ .

### *Ellipsoidal Earth Model*

The centrifugal effect of the earth's rotation causes an equatorial bulge that is the principal departure of the earth from a spherical shape. If the earth behaved like a fluid and there were no convective fluid motions, then it would be in hydrostatic equilibrium and the earth would assume the shape of an ellipsoid of revolution also called the *spheroid*.



parameter	description	formula	value/unit (WGS84)
$GM_e$			$3.986004418 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$
$a$	equatorial radius	-	6378137 m
$c$	polar radius	-	6356752.3 m
$\omega$	rotation rate	-	$7.292115 \times 10^{-5} \text{ rad s}^{-1}$
$f$	flattening	$f = (a - c)/a$	1/298.257223560
$\theta_g$	geographic latitude	-	-
$\theta$	geocentric latitude	-	-

The formula for an ellipse in Cartesian co-ordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1 \quad (3)$$

where the  $x$ -axis is in the equatorial plane at zero longitude (Greenwich), the  $y$ -axis is in the equatorial plane and at  $90^\circ\text{E}$  longitude and the  $z$ -axis points along the spin-axis. The formula relating  $x$ ,  $y$ , and  $z$  to geocentric latitude and longitude is

$$\begin{aligned} x &= r \cos \theta \cos \phi \\ y &= r \cos \theta \sin \phi \\ z &= r \sin \theta \end{aligned} \quad (4)$$

Now we can re-write the formula for the ellipse in polar co-ordinates and solve for the radius of the ellipse as a function of geocentric latitude.

$$r = \left( \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{c^2} \right)^{-1/2} \cong a(1 - f \sin^2 \theta) \quad (5)$$

Before satellites were available for geodetic work, one would establish latitude by measuring the angle between a local plumb line and an external reference point such as Polaris. Since the local plumb line is perpendicular to the spheroid (i.e., local flattened surface of the earth), it points to one of the foci of the ellipse. The conversion between geocentric and geographic latitude is straightforward and its derivation is left as an exercise. The formulas are

$$\tan \theta = \frac{c^2}{a^2} \tan \theta_g \quad \text{or} \quad \tan \theta = (1 - f)^2 \tan \theta_g \quad (6)$$

(Example: What is the geocentric latitude at a geographic latitude of  $45^\circ$ . The answer is  $\theta = 44.8^\circ$  which amounts to a 22 km difference in location!)

#### *Flattening of the Earth by Rotation*

Suppose the earth is a rotating, self gravitating ball of fluid in hydrostatic equilibrium. Then density will increase with increasing depth and surfaces of constant pressure and density will coincide. The surface of the earth will be one of these equipotential surfaces and it has a potential  $U_o$ .

$$U_o = V(r, \theta) - \frac{1}{2} \omega^2 r^2 \cos^2 \theta \quad (7)$$

where the second term on the right side of equation (7) is the change potential due to the rotation of the earth at a frequency  $\omega$ . The potential due to an ellipsoidal earth in a non-rotating frame can be expressed as

$$V = -\frac{GM_e}{r} \left[ 1 - J_1 \frac{a}{r} P_1(\theta) - J_2 \left( \frac{a}{r} \right)^2 P_2(\theta) - \dots \right] \quad (9)$$

The center of the co-ordinate system is selected to coincide with the center of mass so, by definition,  $J_1$  is zero. For this model, we keep only  $J_2$  (dynamic form factor or "jay two" =  $1.08 \times 10^{-3}$ ) so the final reference model is

$$V = -\frac{GM_e}{r} + \frac{GM_e J_2 a^2}{2r^3} (3 \sin^2 \theta - 1) \quad (10)$$

This parameter  $J_2$  is related to the principal moments of inertia of the earth by MacCullagh's formula. For a complete derivation see Stacey, p. 23-35. Let  $C$  and  $A$  be the moments of inertia about the spin axis and equatorial axis respectively. For example

$$C = \int_V (x^2 + y^2) dm \quad (11)$$

After a lot of algebra one can derive a relationship between  $J_2$  and the moments of inertia.

$$J_2 = \frac{C - A}{Ma^2} \quad (12)$$

In addition, if we know  $J_2$ , we can determine the flattening. This is done by inserting equation (10) into equation (7) and noting that the value of  $U_o$  is the same at the equator and the pole. Solving for the polar and equatorial radii that meet this constraint, one finds a relationship between  $J_2$  and the flattening.

$$f = \frac{a - c}{a} = \frac{3}{2} J_2 + \frac{1}{2} \frac{a^3 \omega^2}{GM} \quad (13)$$

Thus if we could somehow measure  $J_2$ , we would know quite a bit about our planet.

#### *Measurement of $J_2$*

Just as in the case of measuring the total mass of the earth, the best way to measure, is to monitor the orbit of an artificial satellite. In this case we measure the precessional period of the inclined orbit plane. To second degree, external potential is

$$V = -\frac{GM_e}{r} + \frac{GM_e J_2 a^2}{2r^3} (3\sin^2\theta - 1) \quad (14)$$

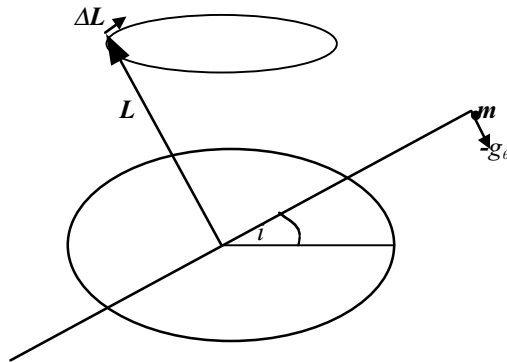
The force acting on the satellite is  $-\nabla V$ .

$$\mathbf{g} = -\frac{\partial V}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} - \frac{1}{r \cos\theta} \frac{\partial V}{\partial \phi} \hat{\phi} \quad (15)$$

If we were out in space, the best way to measure  $J_2$  would be to measure the  $\theta$ -component of the gravity force.

$$g_\theta = \frac{1}{r} \frac{\partial V}{\partial \theta} = -\frac{3GM_e a^2 J_2}{r^4} \sin\theta \cos\theta \quad (16)$$

This component of force will apply a torque to the orbital angular momentum and it should be averaged over the orbit. Consider the diagram below.



For a prograde orbit, the precession  $\omega_p$  is retrograde; that is opposite to the earth's spin direction. The complete derivation is found in Stacey [1977, p. 76] and the result is

$$\frac{\omega_p}{\omega_s} = \frac{-3a^2}{2r^2} J_2 \cos i \quad (17)$$

where  $i$  is the inclination of the satellite orbit with respect to the equatorial plane,  $\omega_s$  is the orbit frequency of the satellite and  $\omega_p$  is the precession frequency of the orbit plane in inertial space.

#### *Example - LAGEOS*

As an example, the LAGEOS satellite orbits the earth every 13673.4 seconds at an average radius of 12,265 km, and an inclination of  $109.8^\circ$ . Given the parameters in the table above and  $J_2 = 1.08 \times 10^{-3}$ , the predicted precession rate is  $0.337^\circ/\text{day}$ . This can be compared with the observed rate of  $0.343^\circ/\text{day}$ . The figure on the next page is an illustration of the LAGEOS satellite.

#### *Hydrostatic Flattening*

Given the radial density structure, the earth rotation rate and the assumption of hydrostatic equilibrium, one can calculate the theoretical flattening of the earth (see Garland, *The Earth's Shape and Gravity*, Permagon, 1977, Appendix 2). This is called the hydrostatic flattening  $f_h = 1/299.5$ . From the table above we have the observed flattening  $f = 1/298.257$  so the actual earth is flatter than the theoretical earth. There are two reasons for this. First, the earth is still recovering from the last ice age when the poles were loaded by heavy ice sheets. When the ice melted polar dimples remained and the glacial rebound of the viscous mantle is still incomplete. Second, the mantle is not in hydrostatic equilibrium because of mantle convection. Finally it should be noted that  $J_2$  is changing with time due to the continual post-glacial rebound. This is called "jay two dot" and it can be observed in satellite orbits as a time variation in the precession rate.



## LAGEOS ASSEMBLY

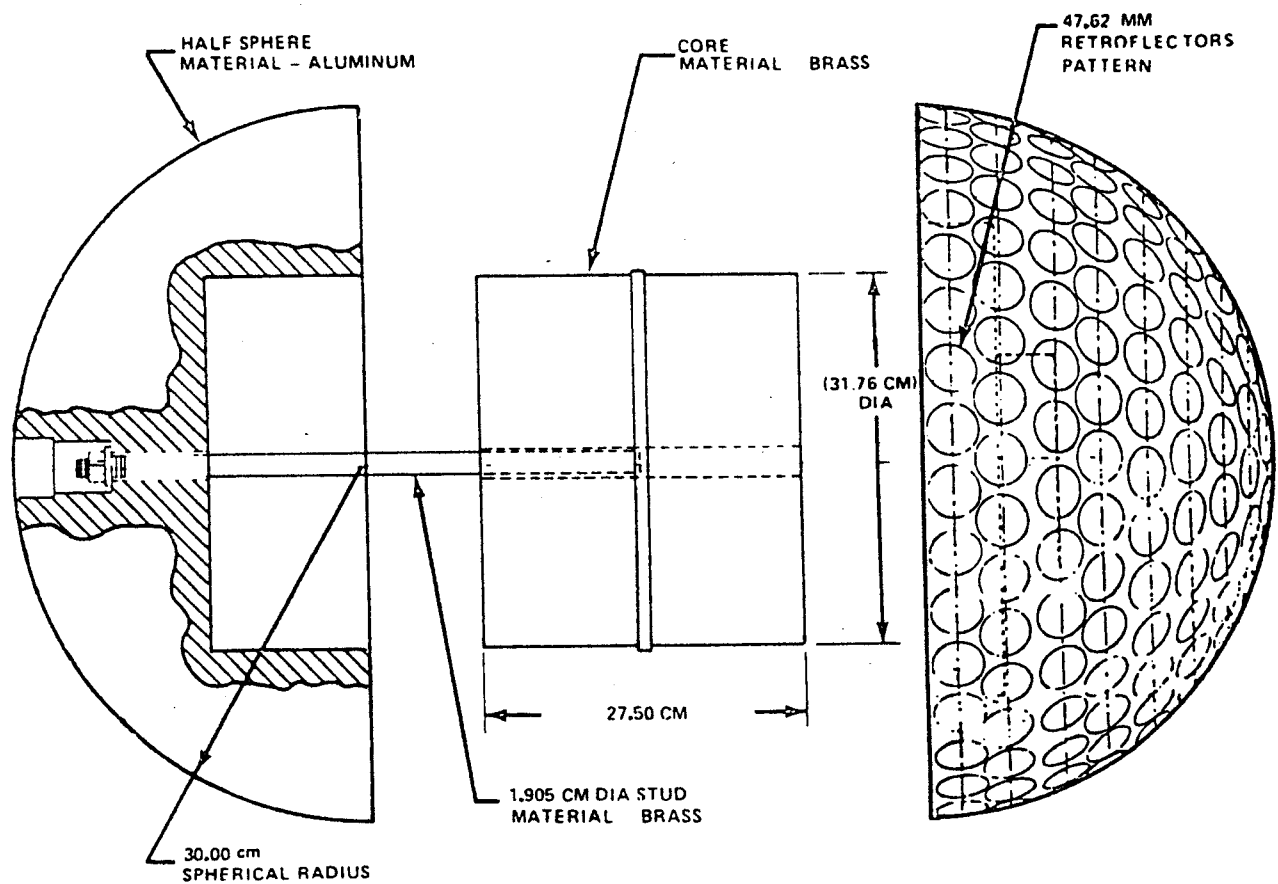


Fig. 1. Structural detail of LAGEOS satellite.

TABLE 1. LAGEOS Orbital Parameters

	Value
Semimajor axis	12265 km
Eccentricity	0.004
Inclination	109.8°
Perigee height	5858 km
Apogee height	5958 km
Perigee rate	-0.214°/d
Node rate	+0.343°/d
Semimajor axis decay rate	-1.1 mm/d
Orbital acceleration	$3 \times 10^{-12}$ m/s