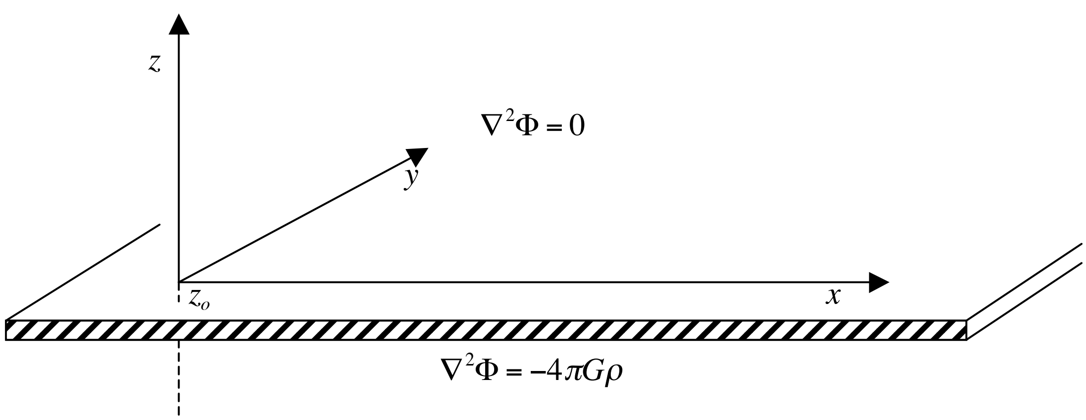
Poisson's Equation in Cartesian Coordinates

As in the lecture on Laplace's equation, we are interested in anomalies due to local structure and will use a flat-earth approximation. However unlike the last lecture, the emphasis is on generating models of the disturbing potential and it derivatives from a 3-D model of the variations in density and topography of the earth. In a following lecture we'll combine this fourier-approach to calculating gravity models with the models for isostasy and flexure to develop a topography to gravity transfer function. Consider the disturbing potential

Φ	=	U	-	U_o	(1)
disturbing potential		total potential		reference potential	

where, in this case, the reference potential comprises the ellipsoidal reference Earth model plus the reference spherical harmonic model. The disturbing potential satisfies Laplace's equation for an altitude, z, above the highest mountain in the area while it satisfies Poisson's equation below this level as shown in the following diagram.



$\Phi(x,y,z)$ -- disturbing potential (total - reference)
G -- gravitational constant
ρ -- density anomaly (total - reference)

First consider a density model consisting of an infinitesimally-thin sheet at a depth z_o having a surface-density of $\sigma(x,y)$ (units of mass per unit area). Later we'll construct a more complicated 3-D structure from a stack of many layers. Poisson's equation is an inhomogeneous second-order partial differential equation in three dimensions.

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2} = -4\pi G\sigma(\mathbf{x})\delta(z-z_o), \quad (2)$$

Six boundary conditions are needed to develop a unique solution. Far from the region, the disturbing potential must go to zero; this accounts for 5 of the boundary conditions

$$\lim_{|x|\to\infty}\Phi = 0, \quad \lim_{|y|\to\infty}\Phi = 0, \quad \lim_{z\to\infty}\Phi = 0 \quad (3)$$

The sixth condition is prescribed by the density model. To solve this differential equation, we'll use the 2-D fourier transform again where the forward and inverse transform are

$$F(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x})e^{-i2\pi(\mathbf{k}\cdot\mathbf{x})}d^2\mathbf{x} \quad (4)$$

$$f(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mathbf{k})e^{i2\pi(\mathbf{k}\cdot\mathbf{x})}d^2\mathbf{k}$$

where $\mathbf{x} = (x, y)$ is the position vector, $\mathbf{k} = (1/\lambda_x, 1/\lambda_y)$ is the wavenumber vector, and $(\mathbf{k} \bullet \mathbf{x}) = k_x x + k_y y$. Fourier transformation reduces Poisson's equation and the surface boundary to

$$-4\pi^2\left(k_x^2+k_y^2\right)\Phi(\mathbf{k},z)+\frac{\partial^2\Phi}{\partial z^2} = -4\pi G\sigma(\mathbf{k})\delta(z-z_o) \quad (5)$$

$$\lim_{z\to\infty}\Phi(\mathbf{k},z) = 0, \quad (6)$$

Next take the fourier transform with respect to z.

$$\pi\left(k_x^2+k_y^2+k_z^2\right)\Phi(\mathbf{k},k_z) = G\sigma(\mathbf{k})e^{-i2\pi k_z z_o} \quad (7)$$

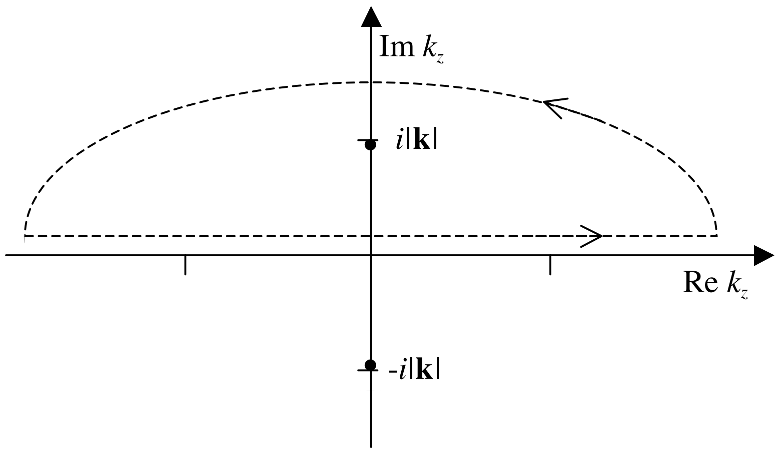
We have used the definition of the delta function $\int_{-\infty}^{\infty}\delta(z-z_o)e^{-i2\pi kz}dz = e^{-i2\pi kz_o}$. Next we solve the differential equation for Φ and take the inverse fourier transform with respect to k_z.

$$\Phi(\mathbf{k},z) = \frac{G\sigma(\mathbf{k})}{\pi}\int_{-\infty}^{\infty}\frac{e^{i2\pi k_z(z-z_o)}}{k_z^2+\left(k_x^2+k_y^2\right)}dk_z \quad (8)$$

Use Calculus of residues to do the integration. The denominator can be factored as follows.

$$k_z^2 + \left(k_x^2 + k_y^2\right) = \left(k_z + i|\mathbf{k}|\right)\left(k_z - i|\mathbf{k}|\right) \tag{9}$$

where $|\mathbf{k}| = \left(k_x^2 + k_y^2\right)^{1/2}$. If $z > z_o$, then to satisfy the boundary condition as $z \to \infty$, one must integrate around the i|k|-pole.



The result is

$$\int_{-\infty}^{\infty} \frac{e^{i2\pi k_z (z - z_o)}}{\left(k_z + i|\mathbf{k}|\right)\left(k_z - i|\mathbf{k}|\right)} dk_z = 2\pi i \frac{e^{-2\pi|\mathbf{k}|(z - z_o)}}{2i|\mathbf{k}|} \tag{10}$$

The solution for the potential for $z > z_o$ is

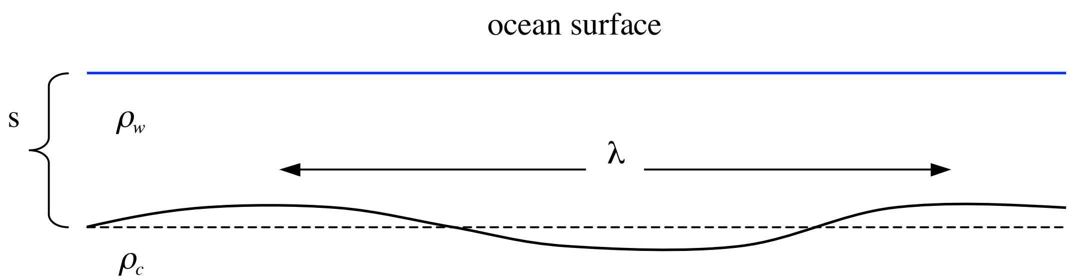
$$\Phi(\mathbf{k}, z) = G\sigma(\mathbf{k}) \frac{e^{-2\pi|\mathbf{k}|(z - z_o)}}{|\mathbf{k}|}. \tag{11}$$

The gravity anomaly is

$$\Delta g(\mathbf{k}, z) = -\frac{\partial \Phi}{\partial z} = 2\pi G\sigma(\mathbf{k}) e^{-2\pi|\mathbf{k}|(z - z_o)}. \tag{12}$$

Example - Gravity due to seafloor topography

Consider topography on the ocean floor $t(\mathbf{x})$ where the maximum amplitude of the topography is much less than the mean ocean depth, s as shown in the following diagram.



Because the topography has low amplitude we can replace the surface density in equation (12) with the topography times the density contrast across the seafloor.

$$\Delta g(\mathbf{k}) = 2\pi G(\rho_c - \rho_w)T(\mathbf{k})e^{-2\pi|\mathbf{k}|s} \tag{13}$$

The result shows that, to a first approximation, the relationship between gravity and topography is linear and isotropic. The ratio of gravity to topography is equal to

$$\frac{\Delta g}{T} = 2\pi G(\rho_c - \rho_w)e^{-2\pi|\mathbf{k}|s} \tag{14}$$

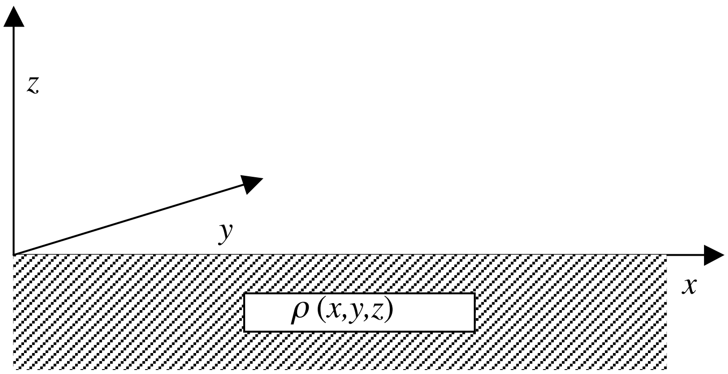
At long wavelength, $|\mathbf{k}| \rightarrow 0$ so the exponential upward continuation term is 1 and the gravity/topography ratio is simply the Bouguer correction term.

$$\frac{\Delta g}{T} = 2\pi G(\rho_c - \rho_w) = 75 mGal/km \tag{15}$$

Suppose the wavelength of the topography is equal to the ocean depth. In this case the exponential, upward-continuation reduces the gravity measured on the ocean surface by a factor of $e^{-2\pi} = 0.0017$. Because of this upward-continuation, topography having wavelength less than the ocean depth become increasingly-difficult to observe in the gravity field at the ocean surface.

Gravity anomaly from 3-D density model

Using this formulation, one can stack, or integrate, these surface density layers over a range of depths to construct the gravity field due to a full 3-D density model.



$$\Phi(\mathbf{k},z) = G\int_{-\infty}^{o}\rho(\mathbf{k},z_o)\frac{e^{-2\pi|\mathbf{k}|(z-z_o)}}{|\mathbf{k}|}dz_o \tag{16}$$

The equivalent expression in the space domain is

$$\Phi(\mathbf{x},z) = G\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{o}\rho(x_o,y_o,z_o)\left[(x-x_o)^2+(y-y_o)^2+(z-z_o)^2\right]^{-1/2}dz_o dy_o dx_o \tag{17}$$

Indeed this is just a statement of the convolution theorem where

$$\Im\left[\left(x^2+y^2+z^2\right)^{-1/2}\right] = \frac{e^{-2\pi|\mathbf{k}|z}}{|\mathbf{k}|}. \tag{18}$$

Computation of geoid height and gravity anomaly

The following table provides the two approaches for calculating geoid height and gravity anomaly from a 3-D density model. The fourier approach involves, 2-D fourier transformation of each layer, adding the upward-continued contribution from each layer, and inverse fourier transformation of the sum. The space-domain approach involves a 3-D convolution of the density model with the 1/*r* (geoid) or z/r^3 (gravity) kernel. For a model with 1024 points in both horizontal directions the fourier approach will be about 50,000 times faster to compute than the space-domain convolution. Moreover the fourier approach will have higher numerical accuracy. because there are fewer additions and subtractions.

space domain	wavenumber domain
$N(\mathbf{x}) = \dfrac{G}{g}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{o}\dfrac{\rho(x_o,y_o,z_o)}{\left[(x-x_o)^2+(y-y_o)^2+z_o^2\right]^{1/2}}dz_o dy_o dx_o$	$N(\mathbf{k}) = \dfrac{G}{g}\int\limits_{-\infty}^{o}\rho(\mathbf{k},z_o)\dfrac{e^{2\pi\lvert\mathbf{k}\rvert z_o}}{\lvert\mathbf{k}\rvert}dz_o$
$\Delta g(\mathbf{x}) = G\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{o}\dfrac{\rho(x_o,y_o,z_o)z_o}{\left[(x-x_o)^2+(y-y_o)^2+z_o^2\right]^{3/2}}dz_o dy_o dx_o$	$\Delta g(\mathbf{k}) = 2\pi G\int\limits_{-\infty}^{o}\rho(\mathbf{k},z_o)e^{2\pi\lvert\mathbf{k}\rvert z_o}dz_o$

Gravity anomaly for a slab of thickness H and a density of ρ_o.

The equation relating gravity to the 3-D density anomaly in the wavenumber domain can be used to calculate the gravity anomaly due to a slab of thickness H and a density of ρ_o. This is used for the Bouguer correction in land gravity surveys. The 3-D density is

$$\rho(\mathbf{x},z) = \begin{bmatrix} \rho_o & -H < z < 0 \\ 0 & z < -H,\ z > 0 \end{bmatrix} \tag{19}$$

The fourier transform of this density is

$$\rho(\mathbf{k},z) = \begin{bmatrix} \delta(k_x)\delta(k_y)\rho_o & -H < z < 0 \\ 0 & z < -H,\ z > 0 \end{bmatrix} \tag{20}$$

The gravity anomaly integral simplifies to

$$\Delta g(\mathbf{k}) = 2\pi G\rho_o\delta(k_x)\delta(k_y)\int\limits_{-H}^{o}e^{2\pi\lvert\mathbf{k}\rvert z_o}dz_o$$

$$= 2\pi G\rho_o\delta(k_x)\delta(k_y)\frac{1}{2\pi\lvert\mathbf{k}\rvert}\left(1 - e^{-2\pi\lvert\mathbf{k}\rvert H}\right). \tag{21}$$

Since only the zero wavenumber component is extracted by the delta function, we expand (23) in a Taylor series about |**k**| and take the limit as $\lvert\mathbf{k}\rvert \to 0$.

$$\lim_{\lvert\mathbf{k}\rvert\to 0}\frac{1}{2\pi\lvert\mathbf{k}\rvert}\left[1 - 1 + 2\pi\lvert\mathbf{k}\rvert H - \frac{(2\pi\lvert\mathbf{k}\rvert H)^2}{2!} + \ldots\right] = H \tag{22}$$

The result in the wavenumber domain is

$$\Delta g(\mathbf{k}) = 2\pi G \rho_o \delta(k_x)\delta(k_y) H. \tag{23}$$

The inverse fourier transform provides the gravity field due to an infinite slab

$$\Delta g(\mathbf{x}) = 2\pi G \rho_o H. \tag{24}$$

Bouguer gravity anomaly

Over the ocean one measures the total acceleration of gravity and subtracts the International Gravity Formula (IGF) to obtain free-air gravity anomaly. Indeed, the free-air anomaly is defined on the geoid which is closely-approximated by the ocean surface. Therefore no corrections are needed for marine gravity measurements.

In contrast, over the land one measures total gravitational acceleration at some elevation h above the geoid; assume this elevation is known from leveling. To reduce these gravity measurements to the geoid, two corrections are commonly applied.

(1) The free-air correction accounts for the decrease in gravity because the observation point is further from the center of the Earth.

(2) The Bouguer correction uses the infinite-slab approximation to account for the gravitational attraction of the rock between the measurement point and the geoid. Note unless the topography is very flat over a large area, this infinite-slab approximation may not be very accurate and a more accurate terrain correction should be applied.

$$\underbrace{\Delta g_B}_{\text{Bouguer gravity}} = \underbrace{g_t}_{\text{measured gravity}} - \underbrace{2\pi G \rho_o h}_{\substack{\text{slab correction} \\ (-0.1118\ \text{mGal/m}) \\ (\rho_o = 2670\ \text{kg m}^{-3})}} + \underbrace{\frac{2GM_e}{R_e^3} h}_{\substack{\text{free-air correction} \\ (0.3086\ \text{mGal/m})}} - \underbrace{\gamma_o(\theta)}_{\text{International Gravity Formula}}$$