The Wine Cellar Problem

Periodic Heating of the Surface of the Earth

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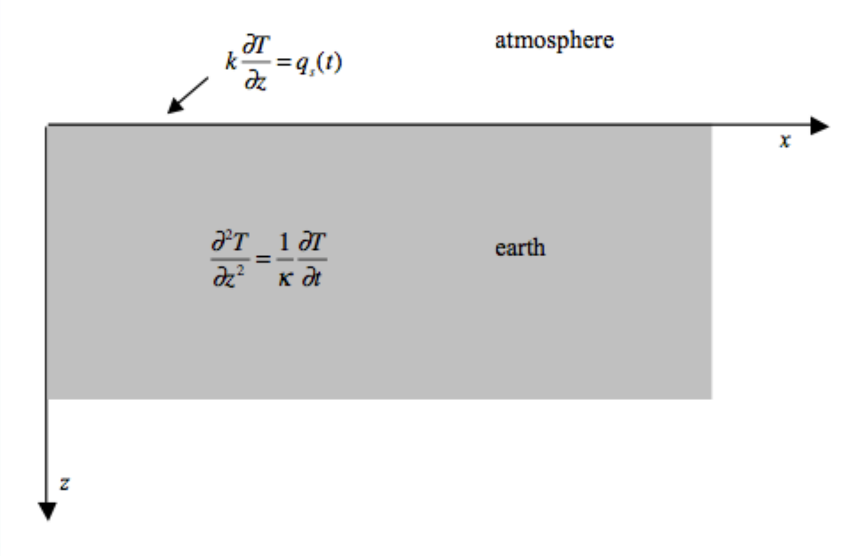
The Problem

- Relate the temperature anomaly at any time and depth to the Fourier transform of the surface heat flux

- What constants determine the attenuation depth?

- What is the attenuation depth of the periodic temperature variations due to the diurnal cycle, annual cycle, and glacial cycle?



Assumptions and Conditions

- For this problem

-we are considering land as a semi infinite homogeneous half space

-land has constant thermal properties

-the heat flux is given as

$$q_s(t) = k\frac{\delta T}{\delta z}$$

-the heating of the Earth is given as

$$\frac{1}{\kappa}\frac{\delta T}{\delta t} = k\frac{\delta^2 T}{\delta z^2}$$

Solution from Heat Flux

- Fourier Transform of heat flux gives us q in the wave domain

$$-k\frac{\delta T}{\delta z} = Ae^{-i\omega t}$$

$$\frac{\delta T}{\delta z} = A^o e^{-i\omega t}$$

- And integrating gives an expression for T

$$T(z,t) = A^o e^{-i\omega t} f(z) + T_o$$

- Using the condition for heating of the Earth, we can find f(z)

$$A^o e^{-i\omega t}\frac{\delta^2 f}{\delta z^2} - \frac{1}{\kappa}A^o f(z) - i\omega e^{-i\omega t} = 0$$

$$A^o e^{-i\omega t}(\frac{\delta^2 f}{\delta z^2} + \frac{i\omega}{\kappa} f(z)) = 0$$

- We have an ODE where the solution for

f(z) is $$ae^{\sqrt{\frac{i\omega}{\kappa}}z} + b^{-\sqrt{\frac{i\omega}{\kappa}}z}$$

- Using the part of f(z) that decays we get

$$T(z,t) = T_o + A' e^{-\sqrt{\frac{\omega}{2\kappa}}z} e^{i(\sqrt{\frac{\omega}{2\kappa}}z + \omega t)}$$

- To find A' we can plug back into our heat flux condition, with z=0 (surface)

$$Ae^{-i\omega t} = -kA'\left(\frac{(i-1)}{\sqrt{2}}\sqrt{\frac{\omega}{\kappa}}\right)e^{(i-1)\sqrt{\frac{\omega}{2\kappa}}(0) - i\omega t}$$

Heat Flux Solution

- From the heat flux derivation we get

$$T(z,t) = T_o + \frac{A}{k}\sqrt{\frac{\kappa}{\omega}}e^{-i\frac{\pi}{4}}e^{-\sqrt{\frac{\omega}{2\kappa}}z}e^{(\sqrt{\frac{\omega}{2\kappa}}z+\omega t)}$$

Solution from Temperature

- We could also have done a temperature based derivation, given:

$$T_s = T_o + \Delta T \sin(\omega t)$$

- This is done in the text, but basically, taking temperature at the surface as a periodic function, the final solution comes out as

$$T(z,t) = T_o + \Delta T e^{(z\sqrt{\frac{\omega}{2\kappa}})} \sin(\omega t - z\sqrt{\frac{\omega}{2\kappa}})$$

Breaking down the Solution

$$T(z,t) = T_o + \Delta T e^{(z\sqrt{\frac{\omega}{2\kappa}})} \sin(\omega t - z\sqrt{\frac{\omega}{2\kappa}})$$

Breaking down the Solution

$$T(z,t) = T_o + \Delta T e^{(z\sqrt{\frac{\omega}{2\kappa}})} \sin(\omega t - z\sqrt{\frac{\omega}{2\kappa}})$$

Mean Temperature

Breaking down the Solution

$$T(z,t) = T_o + \Delta T e^{(z\sqrt{\frac{\omega}{2\kappa}})} \sin(\omega t - z\sqrt{\frac{\omega}{2\kappa}})$$

Mean Temperature

Forcing Amplitude

Breaking down the Solution

$$T(z,t) = T_o + \Delta T e^{(z\sqrt{\frac{\omega}{2\kappa}})} \sin(\omega t - z\sqrt{\frac{\omega}{2\kappa}})$$

Mean Temperature

Forcing Amplitude

Dampening with Depth

Breaking down the Solution

$$T(z,t) = T_o + \Delta T e^{(z\sqrt{\frac{\omega}{2\kappa}})} \sin(\omega t - z\sqrt{\frac{\omega}{2\kappa}})$$

Mean Temperature

Forcing Amplitude

Dampening with Depth

Periodic Forcing

Phase shift

Breaking down the Solution

$$T(z,t) = T_o + \Delta T e^{(z\sqrt{\frac{\omega}{2\kappa}})} \sin(\omega t - z\sqrt{\frac{\omega}{2\kappa}})$$

Mean Temperature

Forcing Amplitude

Dampening with Depth

Periodic Forcing

Phase shift

But where does the wine cellar go????

Wine Cellars

- Constant(-ish) temperature year round.
- For ease of calculation, we look at the skin depth, where

$$T(z_s,t) - T_o = \frac{1}{e}[T(0,t) - T_o]$$

- Time doesn't matter--we're only interested in T at peak amplitude (sine term = 1)

$$\Delta T e^{(z_o\sqrt{\frac{\omega}{2\kappa}})} = \frac{1}{e}(\Delta T e^{((0)\sqrt{\frac{\omega}{2\kappa}})})$$

How far to dig...

$$\Delta T e^{\left(z_o\sqrt{\frac{\omega}{2\kappa}}\right)} = \frac{1}{e}\left(\Delta T e^{\left((0)\sqrt{\frac{\omega}{2\kappa}}\right)}\right)$$

$$e^{\left(z_o\sqrt{\frac{\omega}{2\kappa}}\right)} = \frac{1}{e}$$

$$z_o = \sqrt{\frac{2\kappa}{\omega}}$$

Depth of your wine cellar depends on:

κ – material property of local geology

ω – how long you're keeping the wine!

Daily Fluctuations

- $\omega = 2\pi$ rad/86400 s = 7.27×10^{-5} rad/s

	Sandy Soil	Clay Soil	Rock	Old Snow
κ (m^2s^{-1})	0.18×10^{-6}	0.10×10^{-6}	1.43×10^{-6}	0.05×10^{-6}
z_o	0.07 m	0.05 m	0.20 m	0.04 m

- Life is easy if you are going to drink your wine a few days after you buy it

Annual Fluctuations

- ω = 2π rad/31536000 s = 1.99 x 10^{-7} rad/s

	Sandy Soil	Clay Soil	Rock	Old Snow
κ (m^2s^{-1})	0.18 x 10^{-6}	0.10 x 10^{-6}	1.43 x 10^{-6}	0.05 x 10^{-6}
z_o	1.35 m	1.00 m	3.79 m	0.71 m

- Functional wine cellars for everyone!

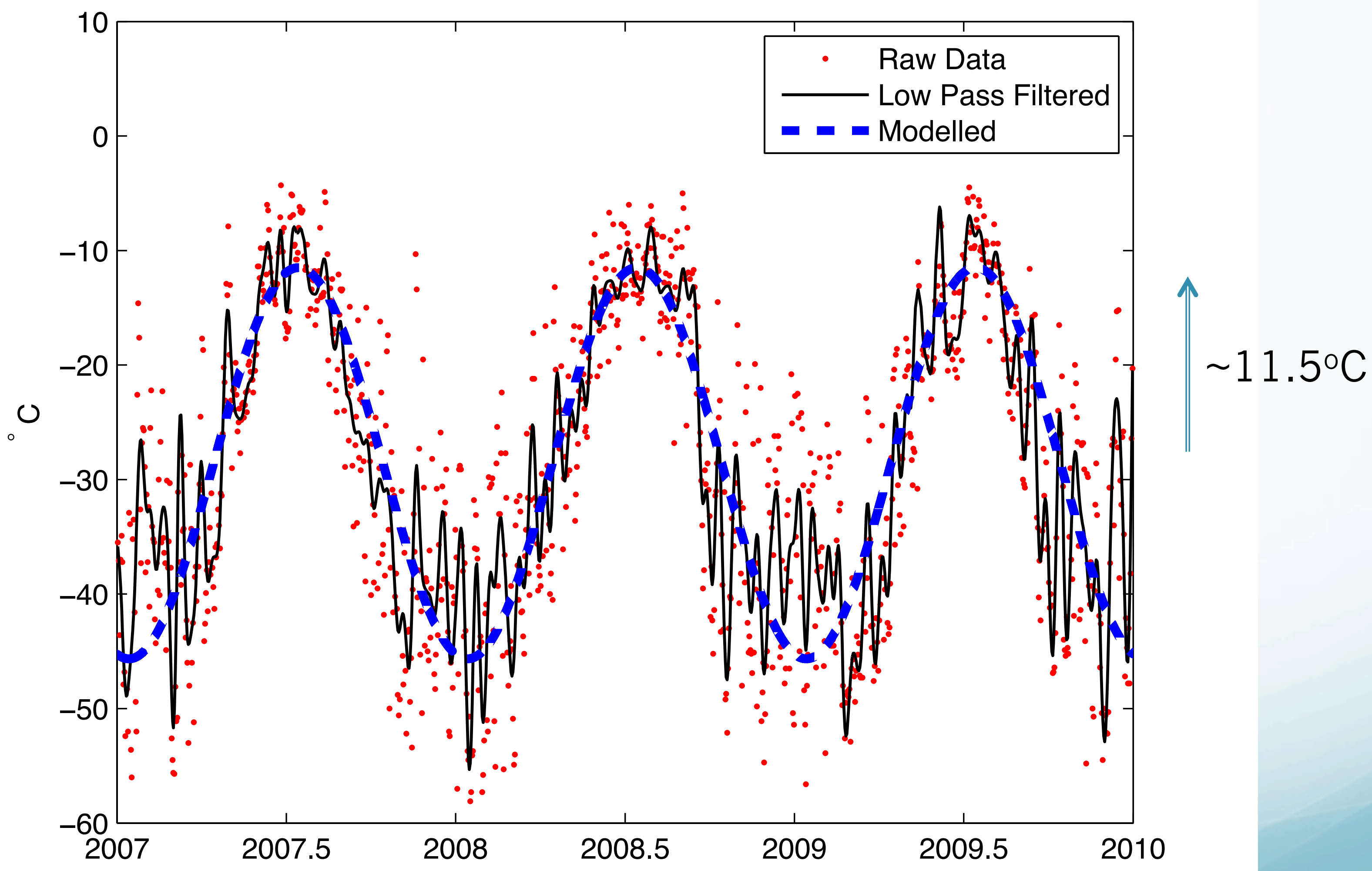
Glacial Fluctuations

- $\omega = 2\pi/31536 \times 10^8$ s $= 1.99 \times 10^{-12}$ rad/s

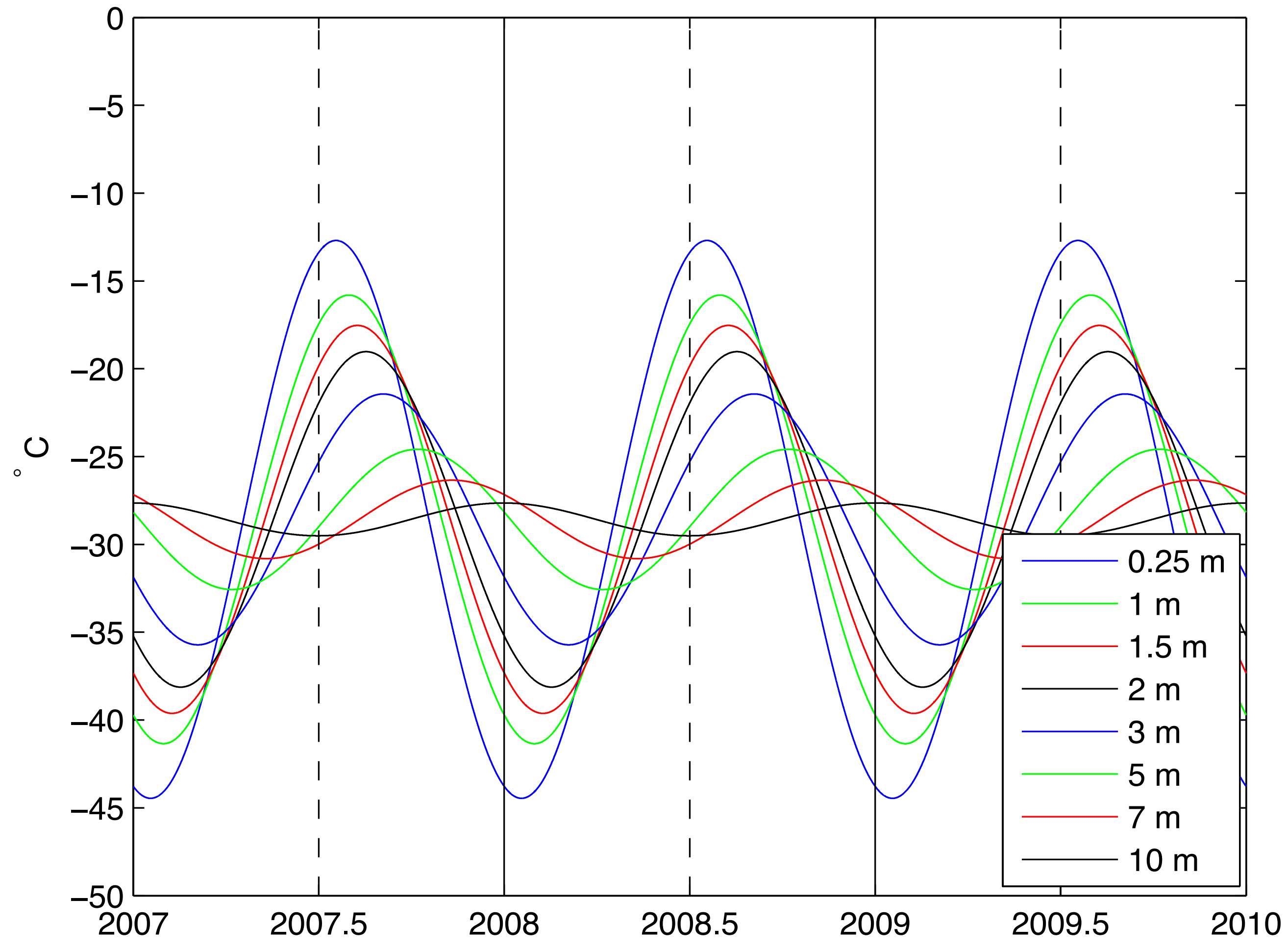
	Sandy Soil	Clay Soil	Rock	Old Snow
κ ($m^2 s^{-1}$)	0.18×10^{-6}	0.10×10^{-6}	1.43×10^{-6}	0.05×10^{-6}
z_o	425 m	317 m	1200 m	224 m

- Borehole thermometry seems practical
- And speaking about ice...

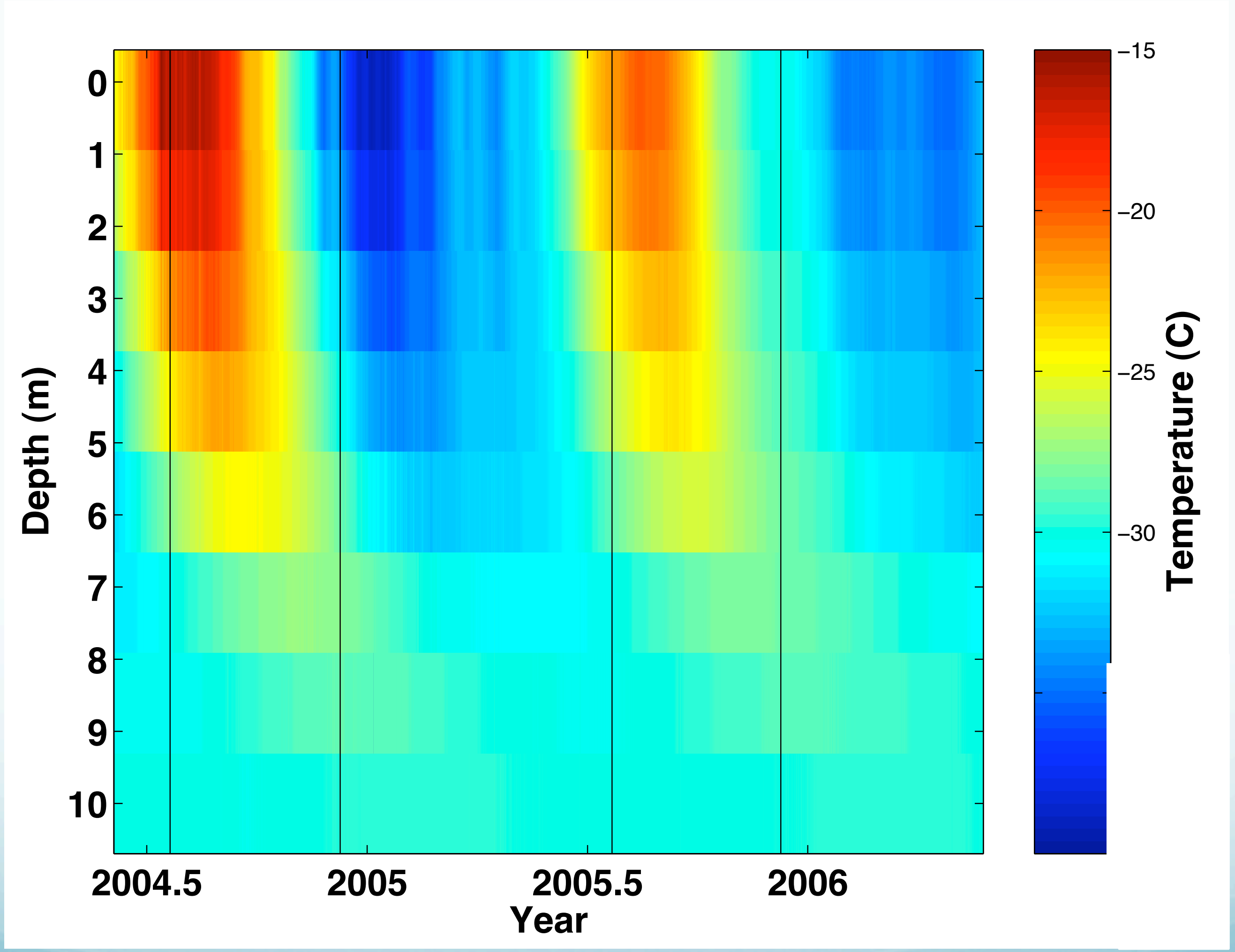
Temperature Forcing at Summit

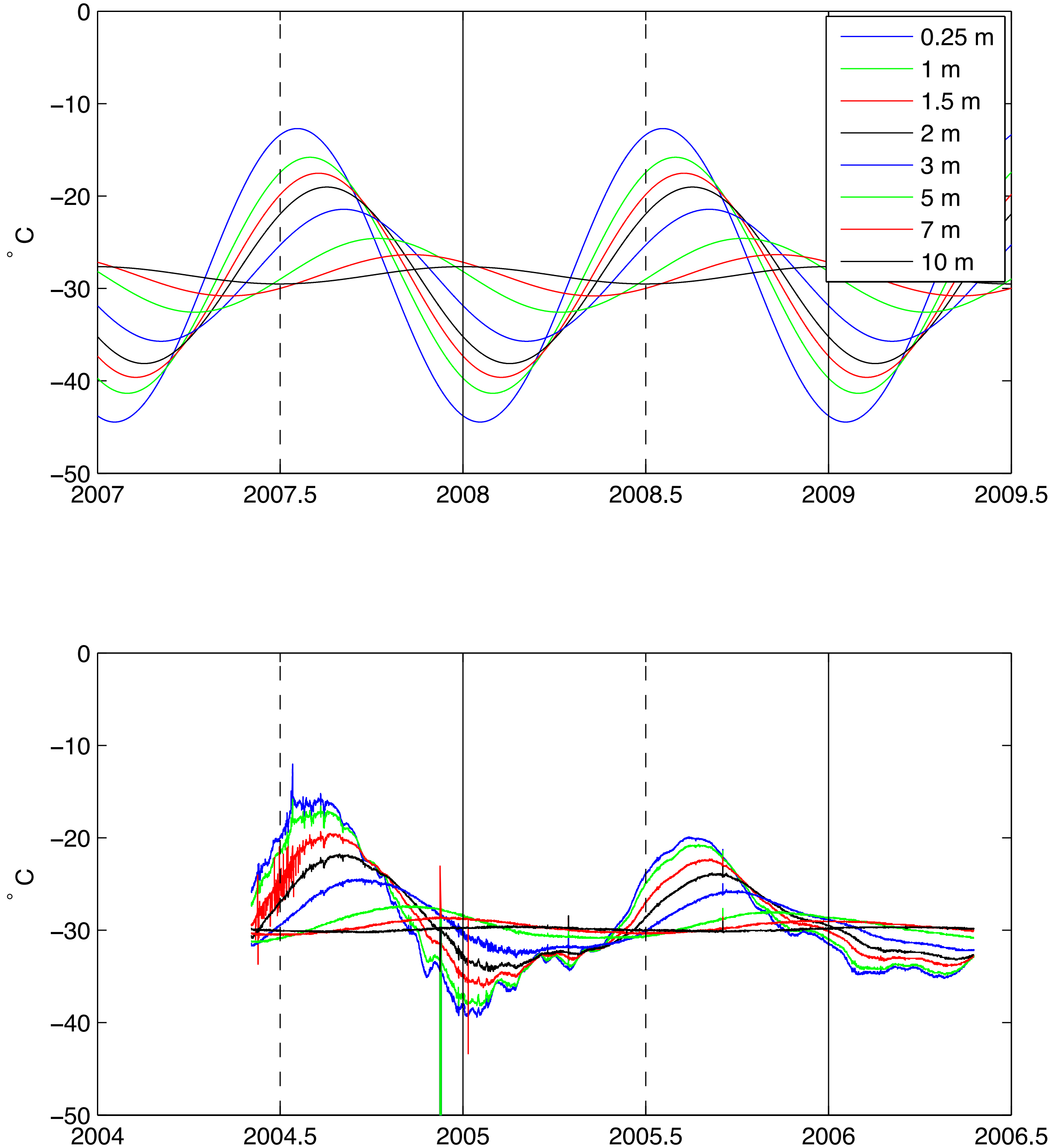


Modeled Temperature at Depth



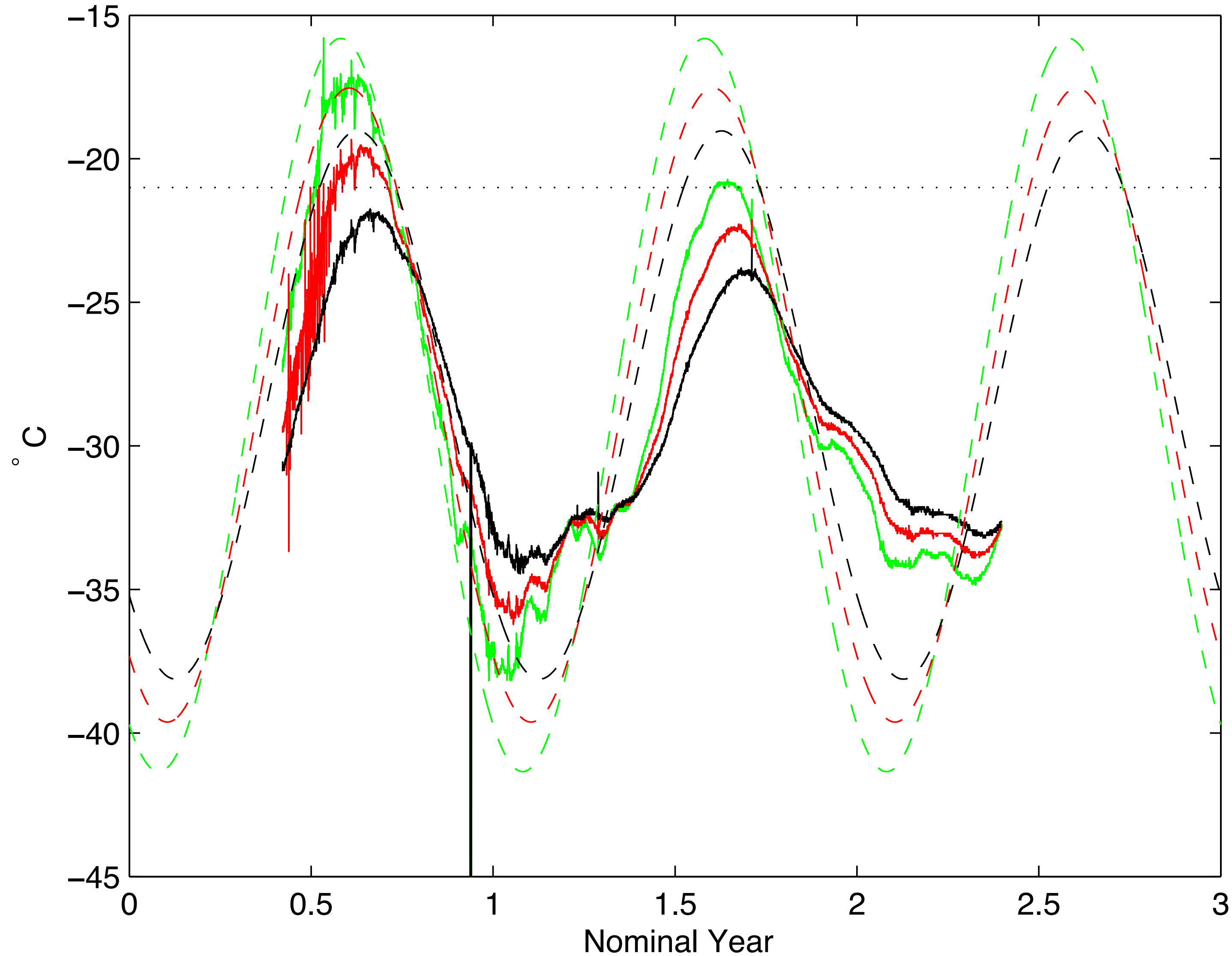
Actual Temperatures at Depth





0
−10
−20
−30
−40
−50
° C
2007
2007.5
2008
2008.5
2009
2009.5
0.25 m
1 m
1.5 m
2 m
3 m
5 m
7 m
10 m
0
−10
−20
−30
−40
−50
° C
2004
2004.5
2005
2005.5
2006
2006.5

Data vs. Model



Conclusions

- Very easy to make a wine cellar that absorbs 63% of annual surface temperature fluctuations
- Sound physical basis for borehole thermometry
- Deviations from heat equation can be surprisingly illuminating