The Wine Cellar Problem

Temperature as a function of depth and time

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SIO234: Geodynamics, Prof. Sandwell

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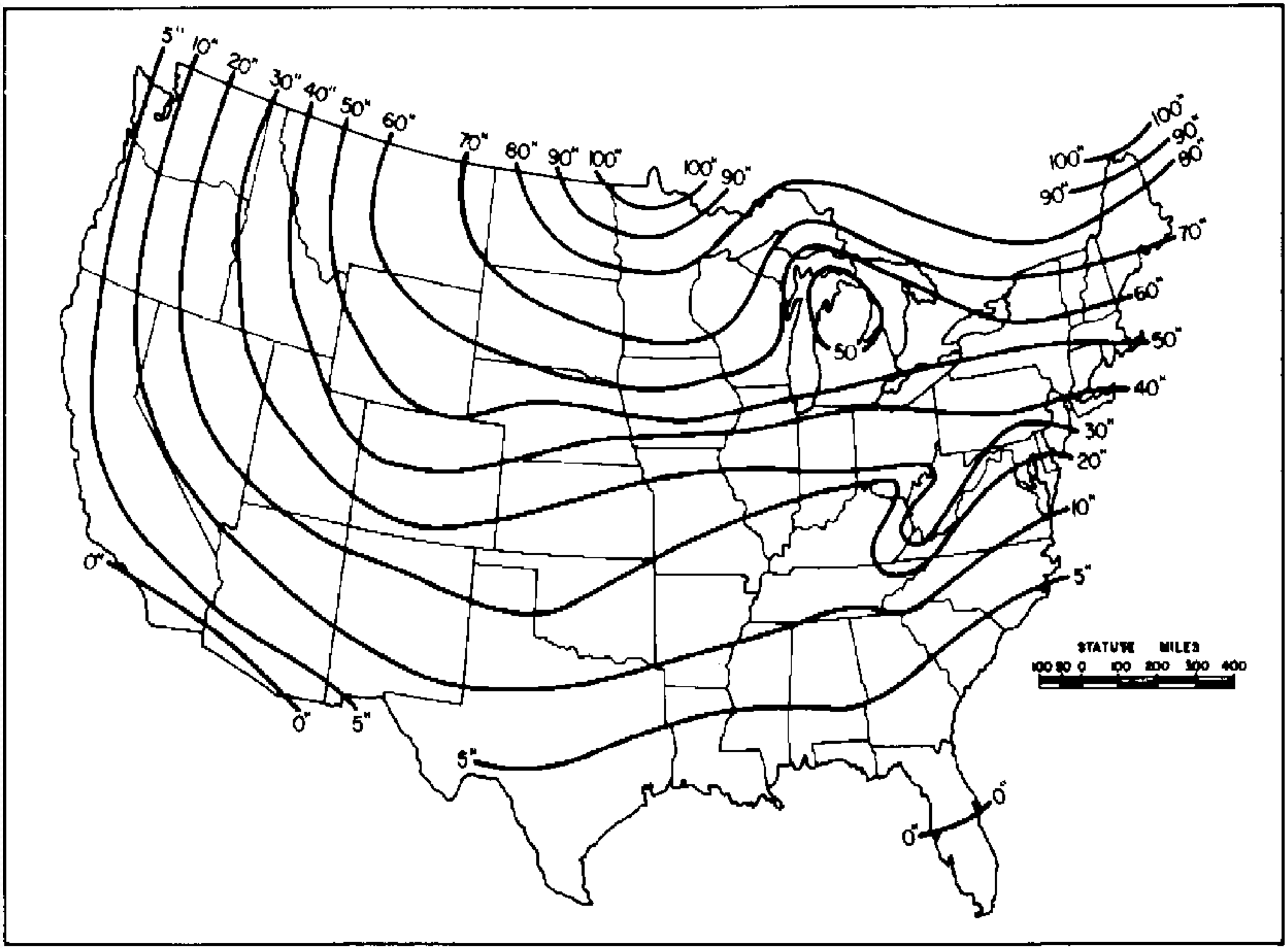
Periodic Heating of the Surface of the Earth

- Why is it the wine cellar problem?
  - Wine ages best when kept at a constant temperature
  - If we understand the heating that occurs at the surface of the Earth, we can choose an appropriate cellar depth

Other Applications

- Geodynamics
  - Determine diffusivity, κ, of rocks and soil
- Structural Engineering
  - Determine required foundation depth to prevent heaving



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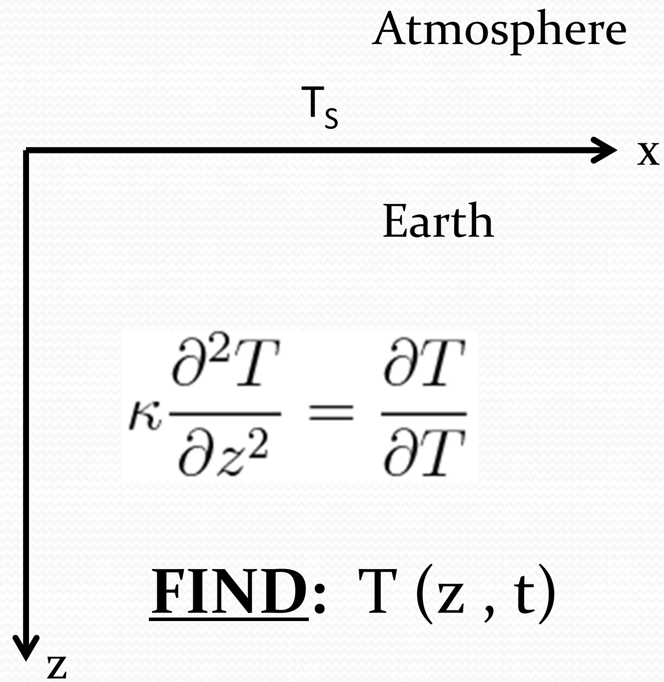
Defining the Problem

- T_S
  - Represents variation in temperature at the surface
  - Assumed periodic
- $T_S = T_o + \Delta T \cos(\omega t)$
  - Initial Condition

    $T(z,0) = T_o$
  - Boundary Conditions

    $T(0,t) = T_S$

    $T(\infty,t) = T_o$



$$\kappa \frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial T}$$

FIND: T (z , t)

Note:

$$\omega = 2\pi f \Rightarrow \tau = \frac{1}{f} = \frac{2\pi}{\omega}$$

Solving for T (z, t)

- Guess

$$T - T_o = A\, e^{kz}\, e^{i\omega t} \qquad [1]$$

$$T = T_o + A\, e^{kz}\, e^{i\omega t}$$

- Partial Differential Equation

$$\kappa \frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial T} \qquad [2]$$

- Must take the corresponding partials of the guess

Solving for T (z, t)

- Derivative as a function of time

$$\frac{\partial T}{\partial t} = Ae^{kz} i\omega e^{i\omega t} \quad [3]$$

- Second derivative as a function of depth

$$\frac{\partial T}{\partial z} = Ake^{kz} e^{i\omega t}$$

$$\frac{\partial^2 T}{\partial z^2} = Ak^2 e^{kz} e^{i\omega t} \quad [4]$$

Solving for T (z, t)

- Plug [3] and [4] into [2]

Such that $\kappa \frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial T}$ becomes

$$\kappa \, A \, k^2 \, e^{kz} \, e^{i\omega t} = A \, e^{kz} \, i\omega \, e^{i\omega t}$$

$$\kappa \, k^2 = i\omega$$

$$k^2 = \frac{i\omega}{\kappa}$$

$\rightarrow$ Therefore,

$$k = \pm\sqrt{\frac{i\omega}{\kappa}}$$

Solving for T (z, t)

- Because temperature must decrease, or *decay*, as depth increases, we can reject the positive root of k such that

$$k = -\sqrt{\frac{i\omega}{\kappa}} \qquad [5]$$

- Note : $i = \frac{(i+1)^2}{2}$

$\rightarrow$ Therefore

$$k = -(i+1)\sqrt{\frac{\omega}{2\kappa}} \qquad [6]$$

Solving for T (z, t)

- Plug k [6] back into the initial guess [1]

$$T - T_o = A\, e^{kz}\, e^{i\omega t}$$

$$T - T_o = Ae^{-(i+1)\sqrt{\frac{\omega}{2\kappa}}z} e^{i\omega t}$$

- This can simplify to

$$T - T_o = Ae^{-\sqrt{\frac{\omega}{2\kappa}}z} e^{i(\omega t - \sqrt{\frac{\omega}{2\kappa}}z)}$$

<u>Note</u>: $e^{i\theta} = \cos\theta + i\sin\theta$

Solving for T (z, t)

- Using $e^{i\theta} = \cos\theta + i\sin\theta$ we get

$$T - T_o = Ae^{-\sqrt{\frac{\omega}{2\kappa}}z}e^{i(\omega t - \sqrt{\frac{\omega}{2\kappa}}z)}$$

$$T - T_o = Ae^{-\sqrt{\frac{\omega}{2\kappa}}z}[cos(\omega t - \sqrt{\frac{\omega}{2\kappa}}z) + isin(\omega t - \sqrt{\frac{\omega}{2\kappa}}z)]$$

Solving for T (z, t)

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Note: Since we are only concerned with the real component of the equation, the imaginary component can be disregarded.

Solving for T (z, t)

- Using boundary condition of $T(0, t) = T_S$

$$T - T_o = Ae^{-\sqrt{\frac{\omega}{2\kappa}}z}[cos(\omega t - \sqrt{\frac{\omega}{2\kappa}}z)]$$

$$A = \Delta T$$

- The temperature variation in the Earth due the periodic surface temperatures is

$$\boxed{T(z,t) = T_o + \Delta T e^{-\sqrt{\frac{\omega}{2\kappa}}z}[cos(\omega t - \sqrt{\frac{\omega}{2\kappa}}z)]}$$

What determines the near surface temperature in the Earth?

$$T(z,t) = \color{red}{T_o} + \Delta T e^{-\sqrt{\frac{\omega}{2\kappa}}z}[cos(\omega t - \sqrt{\frac{\omega}{2\kappa}}z)]$$

- T_o – average surface temperature

What determines the near surface temperature in the Earth?

$$T(z,t) = T_o + \Delta T e^{-\sqrt{\frac{\omega}{2\kappa}}z}[cos(\omega t - \sqrt{\frac{\omega}{2\kappa}}z)]$$

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- ΔT – ½ total temperature range (amplitude)

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- cosine – periodic change in temperature

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- $e^{-\sqrt{\frac{\omega}{2\kappa}}z}$ – decay of temp. variation with depth (κ)
- cosine – periodic change in temperature
- $\sqrt{\frac{\omega}{2\kappa}}z$ – phase shift due to time lag between surface temp. and temp. at depth z

How deep do we put a wine cellar?

- Wine cellar should be put at a depth where temperature fluctuations due to periodic surface changes will be minimal
- Apply concept of *skin depth*
  - Skin Depth, z_o
    - depth at which temperature fluctuation is 1/e of the surface temperature fluctuation
    - $1/e \approx 0.37$, which therefore represents a 63% reduction in temperature fluctuation

Solving for Skin Depth

- When looking at T (z, t), consider only the terms containing z

$$\Delta T e^{-\sqrt{\frac{\omega}{2\kappa}}z}[cos(\omega t - \sqrt{\frac{\omega}{2\kappa}}z)]$$

- We are concerned with the depth, z_o, that is necessary to reduce the maximum surface value fluctuation
- Maximum occurs when

$$cos(\omega t - \sqrt{\frac{\omega}{2\kappa}}z) = 1$$

Solving for Skin Depth

- To determine z_o, set

$$\Delta T e^{-\sqrt{\frac{\omega}{2\kappa}}z}\Big|_{z=z_o} = \frac{1}{e}\Delta T e^{-\sqrt{\frac{\omega}{2\kappa}}z}\Big|_{z=0}$$

$$e^{-\sqrt{\frac{\omega}{2\kappa}}z_o} = e^{-1}$$

$$\sqrt{\frac{\omega}{2\kappa}}z_o = 1$$

$$\boxed{z_o = \sqrt{\frac{2\kappa}{\omega}}}$$

<u>Note</u>: The depth z_o can easily be adjusted to represent a different reduction in surface temperature fluctuation by replacing 1/e with the desired value

i.e. – replace 1/e with 0.10 for depth at which 90% reduction occurs

What constants determine z_o?

$$z_o = \sqrt{\frac{2\kappa}{\omega}}$$

1. $\kappa = \dfrac{k}{\rho c}$ = thermal diffusivity [m²/s]

- k = thermal conductivity [W/m K]
- ρ = density [kg/m³]
- c = specific heat [J/kg K]

- These values are **physical properties** of the local geology

What constants determine z_o?

$$z_o = \sqrt{\frac{2\kappa}{\omega}}$$

2. ω = circular frequency

- Recall $\omega = 2\pi f$ and $\tau = \frac{1}{f} = \frac{2\pi}{\omega}$
- τ = period of concern [sec]
- τ varies depending on what time scale is of concern

What constants determine z_o?

- ω values for different time scales provided in the table below

	Diurnal	Weekly	Annual	Glacial
	1 day	7 days	1 year	40,000 years
τ (sec)	86400	604800	31536000	1.261E+12
f (sec^{-1})	1.157E-05	1.653E-06	3.171E-08	7.927E-13
ω (rad/s^{-1})	7.272E-05	1.039E-05	1.992E-07	4.981E-12

- As τ becomes larger, ω gets smaller
- ω is inversely related to z_o

$\Rightarrow$ **Wine cellar depth must increase as τ increases**

Simplified T (z, t)

- By substituting z_o into T (z, t)

$$T(z,t) = T_o + \Delta T e^{-\frac{z}{z_o}}[cos(\omega t - \frac{z}{z_o})]$$

Related to Class Discussion

- Stated in class that a good approximation of the depth of propagation of a temperature change, L, over a time, t is

$$L = \sqrt{\kappa t}$$

- Let t $= \tau = 2\pi/\omega$

$$L = \sqrt{\frac{\kappa 2\pi}{\omega}} = \sqrt{\frac{2\kappa}{\omega}\pi} = \sqrt{\pi}\sqrt{\frac{2\kappa}{\omega}}$$

$$L = 1.77\sqrt{\frac{2\kappa}{\omega}} = 1.77 z_o$$

$\rightarrow$ It can be seen that the approximate method will quickly yield a result, L, of the same magnitude as z_o

Attenuation Depth based on Soil Properties

$$z_0 = \sqrt{\frac{2K}{\omega}}$$

- Thermal diffusivity – soil property

	Ice	Sandy Soil	Clay Soil	Peat Soil	Rock
K (m^2/s)	1.16×10^{-6}	0.24×10^{-6}	0.18×10^{-6}	0.10×10^{-6}	1.43×10^{-6}

http://apollo.lsc.vsc.edu/classes/met455/notes/section6/2.html

- Frequency – time-dependent

	Diurnal	Annual	Glacial
ω (s^{-1})	7.27×10^{-5}	1.99×10^{-7}	4.98×10^{-12}

Attenuation Depth

- How deep is the attenuation depth or skin depth?
  - Represents depth where surface temperature can only affect by 37%

	Ice	Sandy Soil	Clay Soil	Peat Soil	Rock
Diurnal	0.179 m	0.081 m	0.070 m	0.052 m	0.198 m
Annual	3.412 m	1.552 m	1.344 m	1.002 m	3.789 m
Glacial	682 m	310 m	268 m	200 m	757 m

Wine Cellar Depth

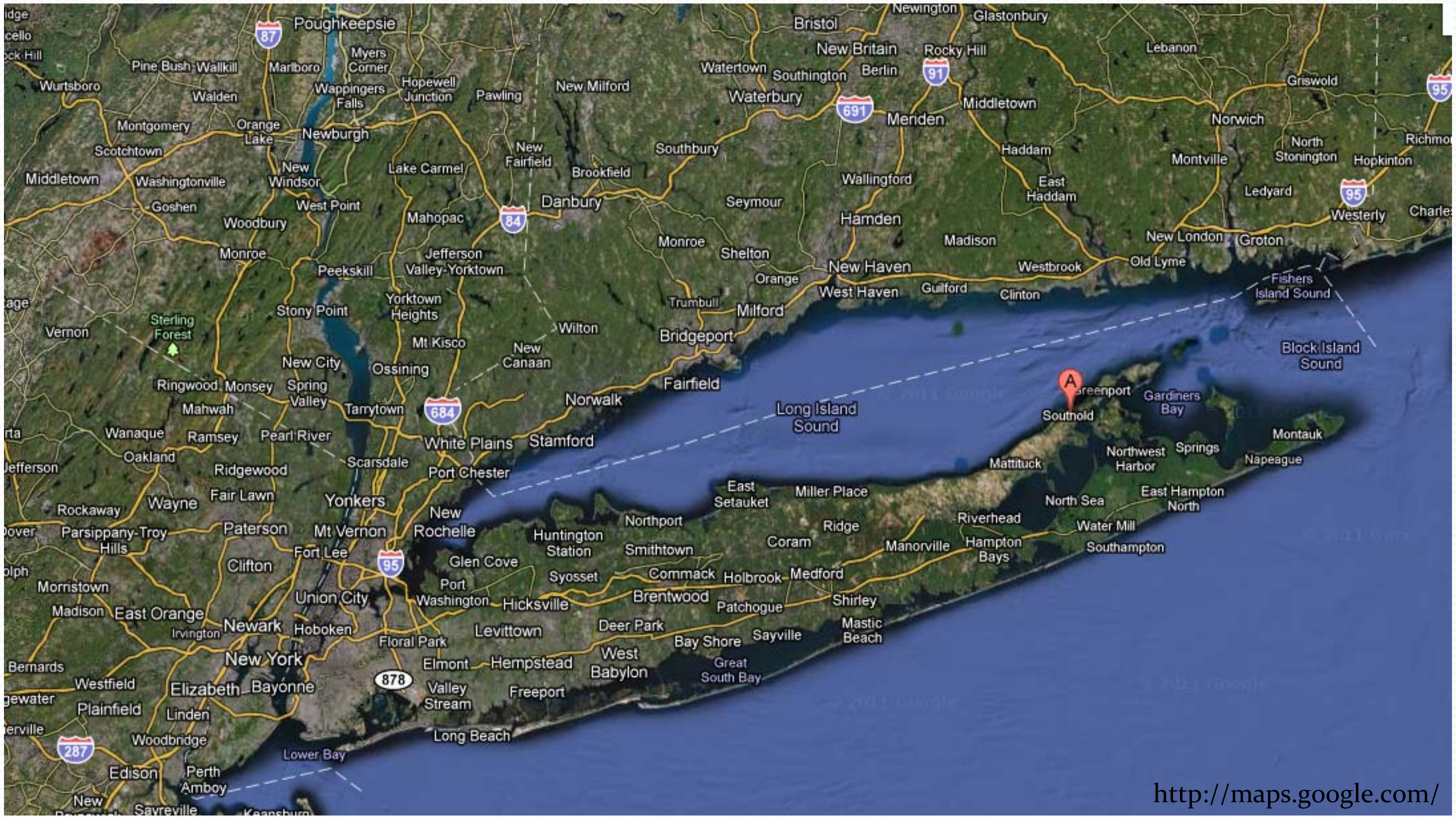
- Attenuation depth doesn't tell us everything we need to know:

$$T(z,t) = T_0 + \Delta T e^{-\frac{z}{z_0}} \cos\left(\omega t - \frac{z}{z_0}\right)$$

- Want cellar to remain unaffected by annual temperature changes
  - $\cos\left(\omega t - \frac{z}{z_0}\right)$ → 1.0 as the maximum
  - Can obtain T(z) relationship which will become steady at a certain dept

Case Study – Long Island, NY

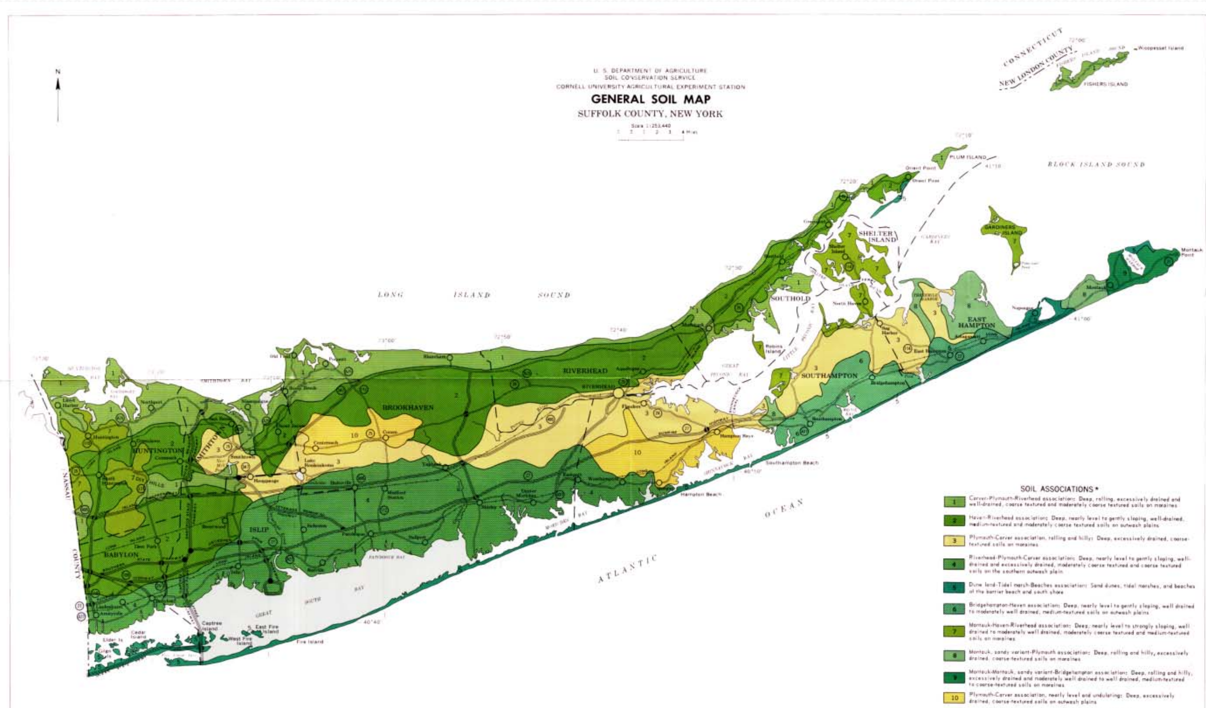
- Suffolk County, New York
  - Active Wine Country!



http://maps.google.com/

Case Study – Long Island, NY

- Soil Type: well-drained, medium-moderately coarse textured soils
  - Sandy Soil: $\kappa = 0.24 \times 10^{-6}$
  - $z_0 = 1.552$ m



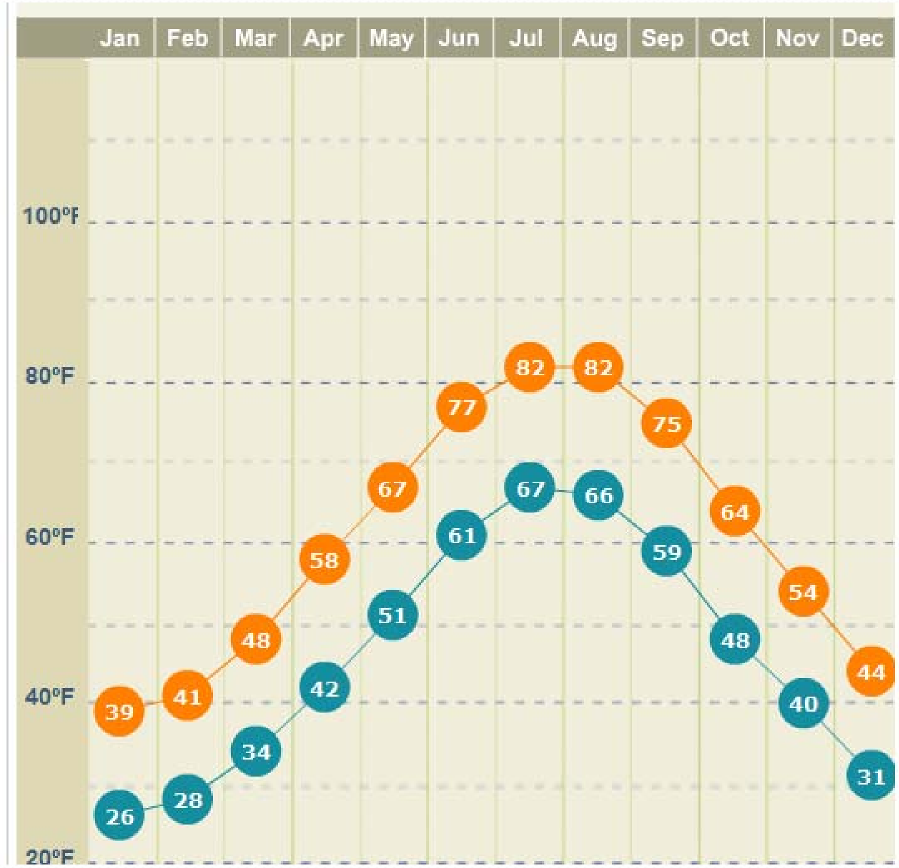
New York Online Soil Survey. Retrieved from: http://soils.usda.gov/survey/online_surveys/new_york/

Case Study – Long Island, NY

- Average Annual Temp
  - $T_o = 53.5°F = 11.9°C$
- Average Range of Temp
  - $\Delta T = 21°F = -6.1°C$
- Surface Temperature:

$T_s(t) = T_o - \Delta T \cos(\omega t)$

$T_s(t) = 11.9 - 6.1 \cos(1.99E\text{-}7 * t)$



http://www.weather.com/weather/wxclimatology/monthly/graph/42689:20

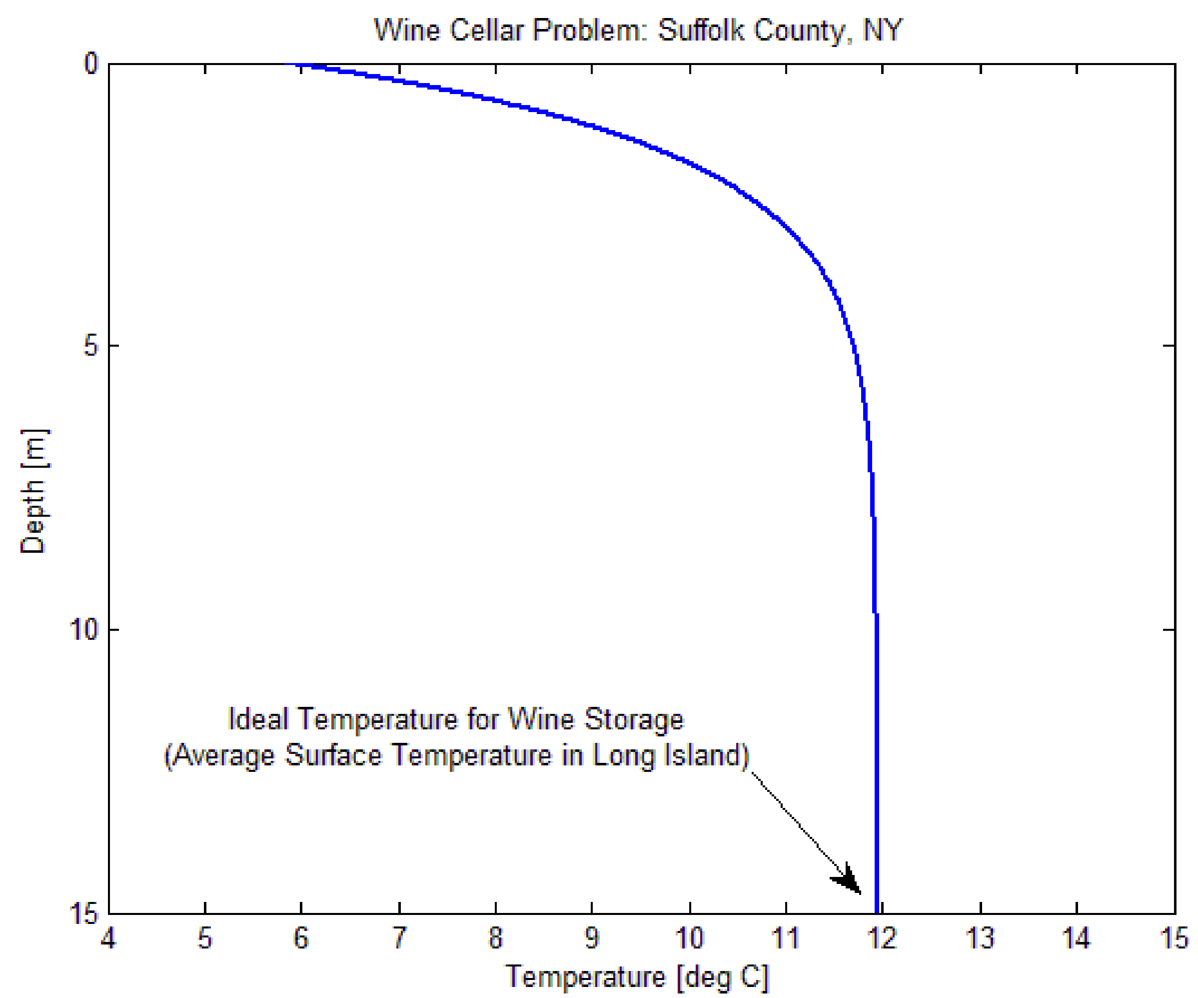
Case Study – Long Island, NY

- Recall:

$$T(z,t) = T_0 + \Delta T e^{-\frac{z}{z_0}} \cos\left(\omega t - \frac{z}{z_0}\right)$$

- Determined: temperature and attenuation parameters
- Maximum effects require cosine term to 1
- Now have:
  - maximum temperature effects along depth

Case Study – Long Island, NY



How accurate is this?

Many assumptions made:

- Soil is homogeneous throughout depths
- Soil thermal properties are accurate
- Water table is sufficiently low
- Heat flow is 1-dimensional

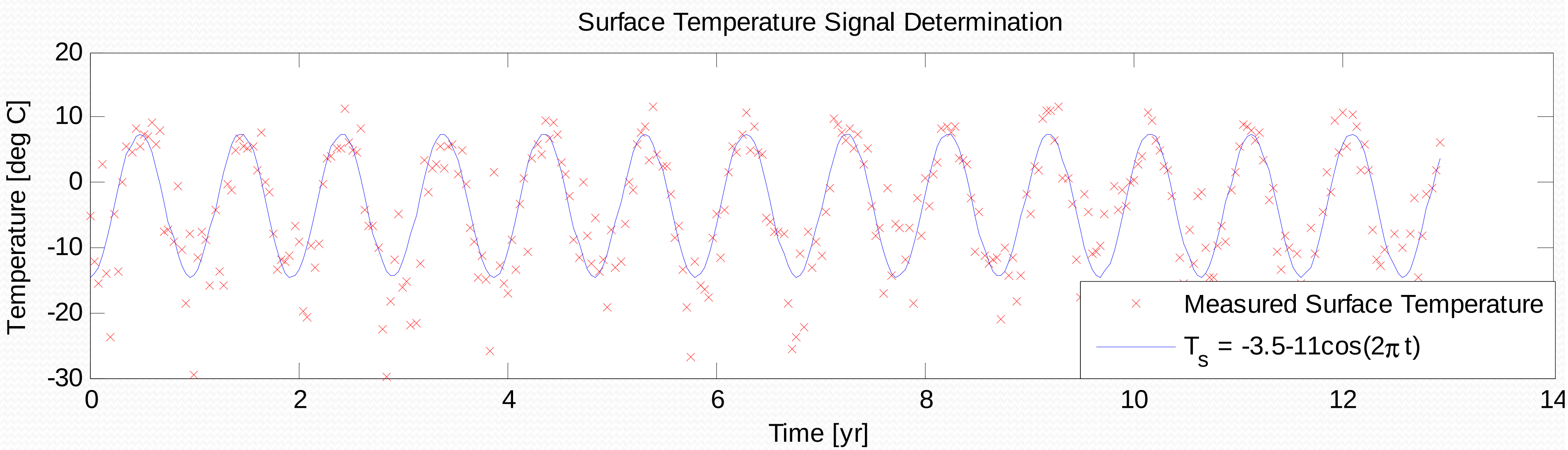
- Cosine accurately represents the annual surface temperature cycle
  - Can look a bit further at this…

Borehole Comparison

- To take a look at the cosine representation
  - Measured borehole-thermistor data
    - Thanks to the National Snow and Ice Data Center!
    - Ilulissat, Greenland
  - Model forcing function as a cosine
  - Compare!
- Remember: Our model is only ever as good as our material (soil) properties!
  - Soil type: Arctic brown soil
  - Assume a diffusivity slightly greater than sandy soil
    - $\kappa = 0.28$

Borehole Comparison

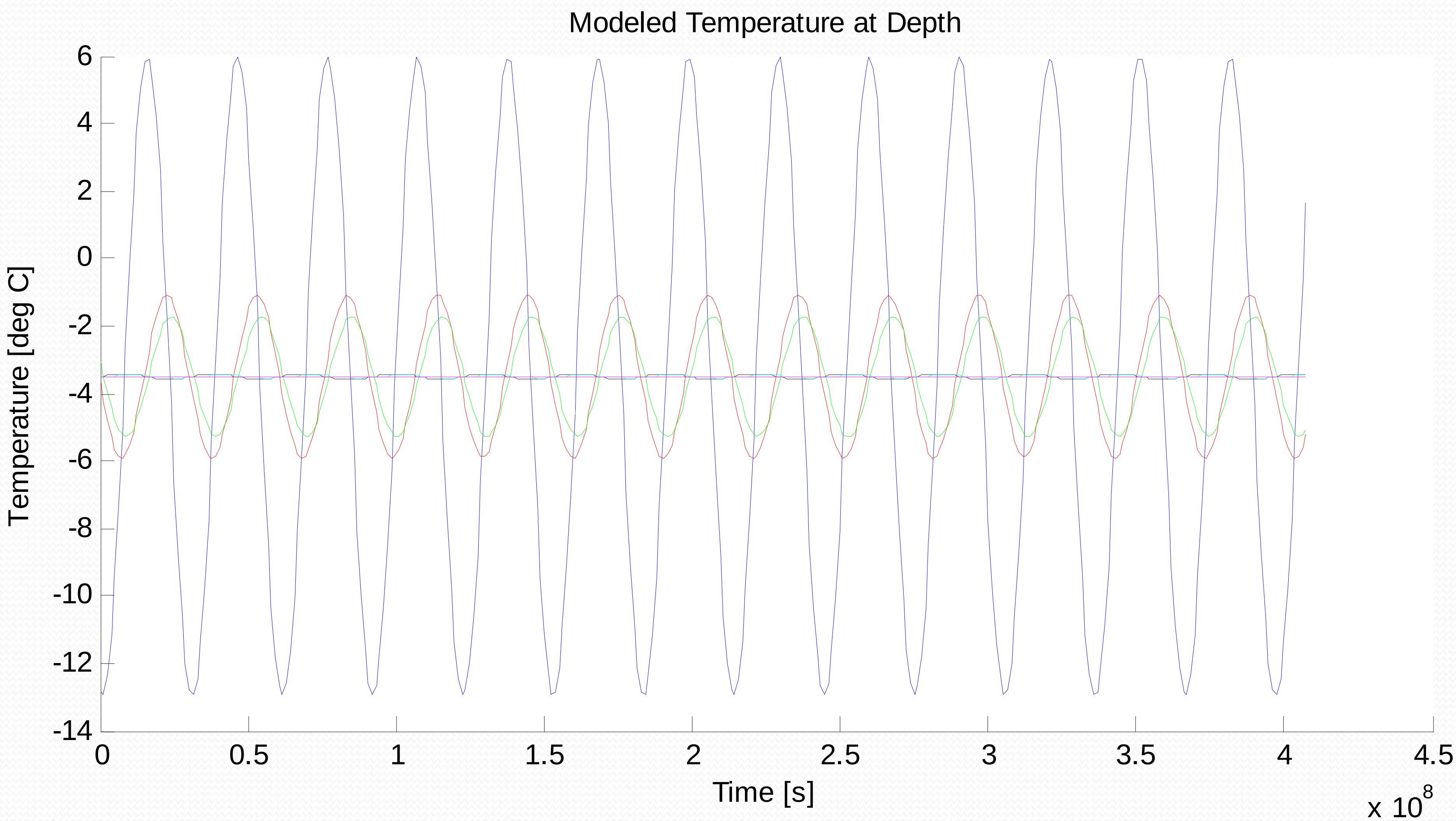
- NSIC provided surface temperature data
- Fit a cosine function!



Borehole Comparison

- Approximate surface forcing function:
  - $T_s(t) = -3.5 - 11 \cos(2\pi t)$ [yr]
  - Attenuation: $z_0 = \sqrt{\frac{2\kappa}{\omega}} = 1.648$ m
  - Depths Sampled:
    - 0.25 m
    - 2.5 m
    - 3.0 m
    - 8.0 m
    - 9.0 m
    - 15.0 m

Borehole Comparison



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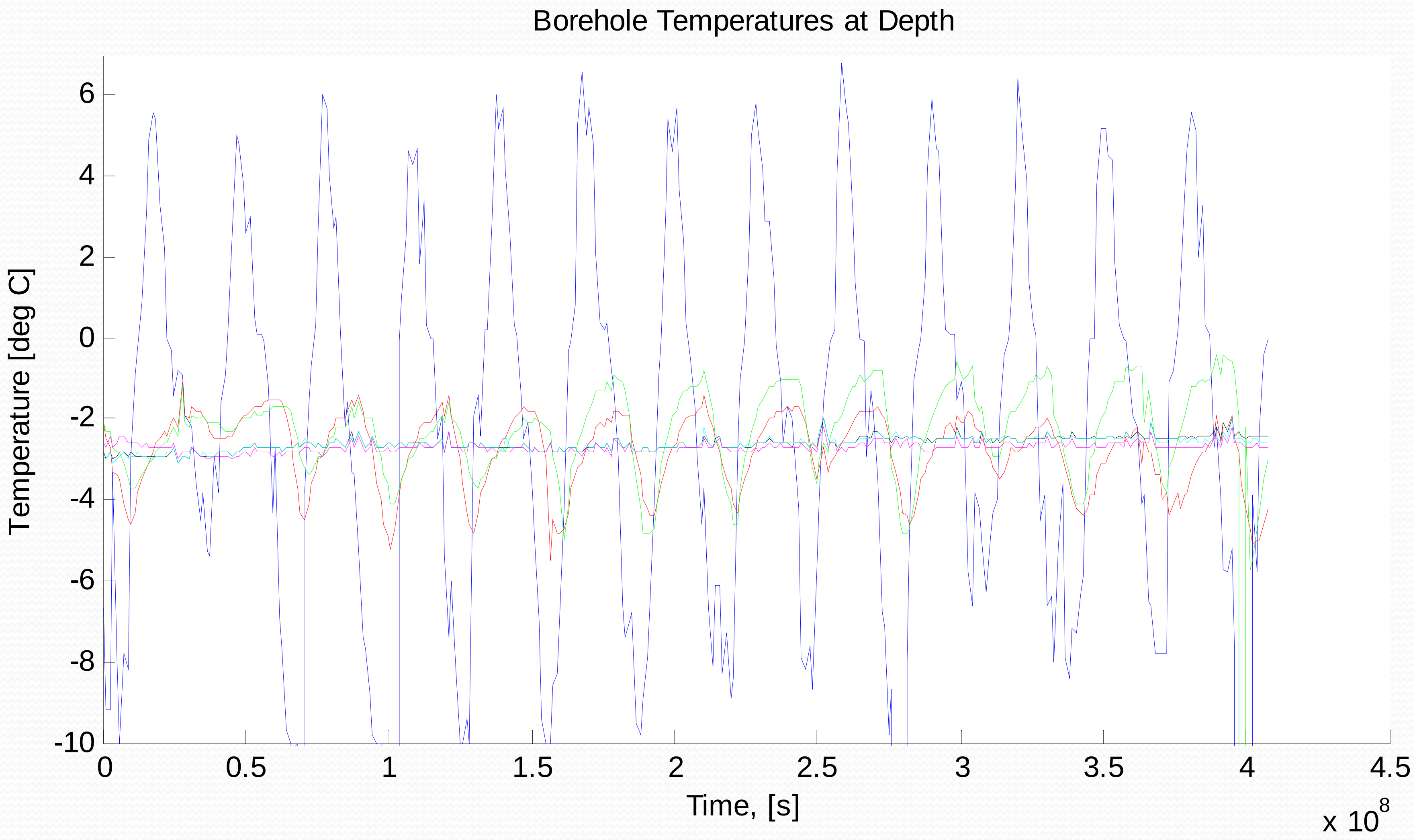
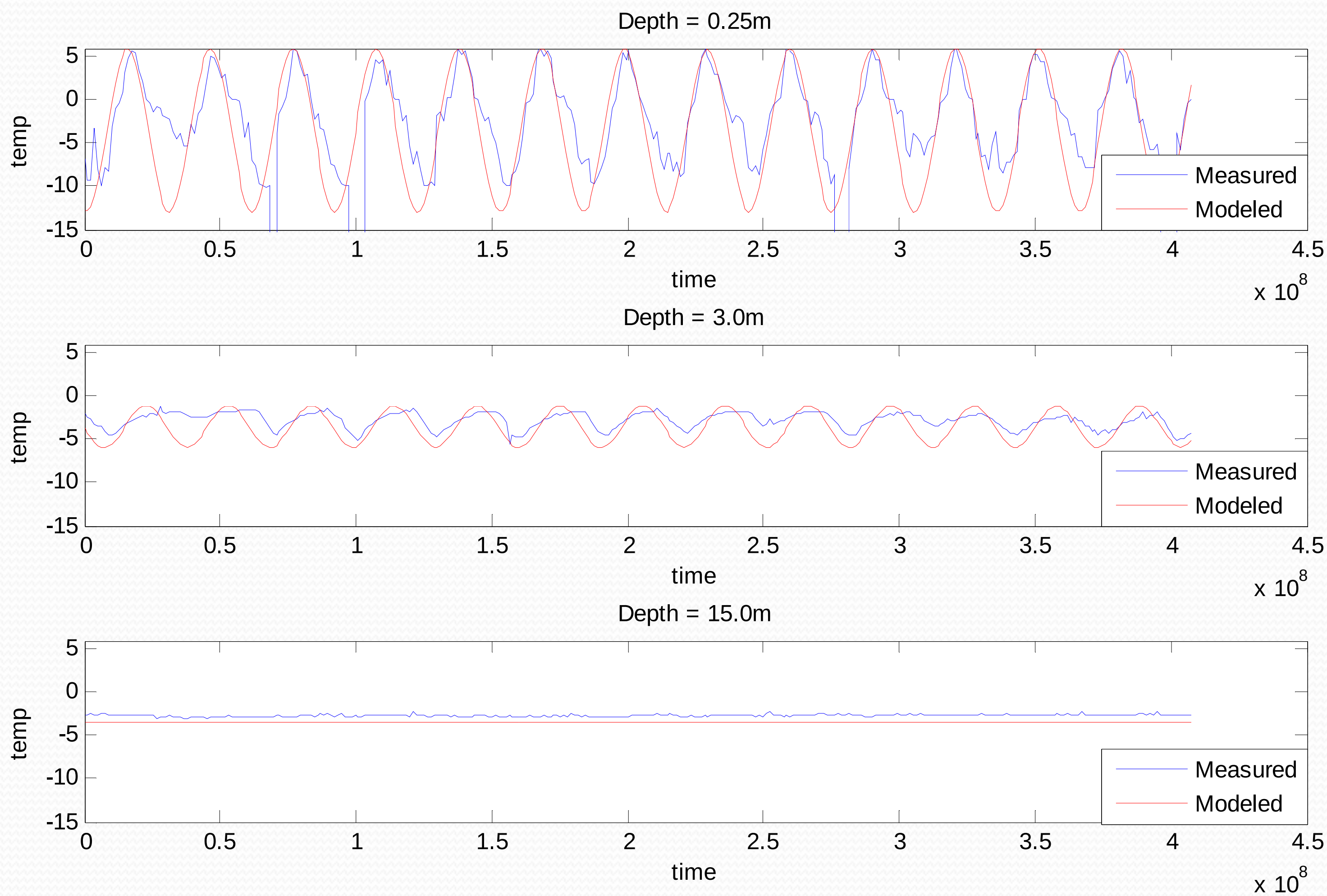


Borehole Comparison



Conclusions

- Wine cellars ~ 8-10m beneath ground in Long Island
- Temperature at depth can be predicted
  - Cosine – a decent approximation to the annual temperature fluctuations
  - Soil properties are important!
    - Even if model at surface is perfect, knowledge of soil at depths is critical.

References

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Questions?

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