The “classic” Wine Cellar Problem

- Time-dependent heat conduction in a 1D half-space
- Surface experiences a (known) temperature or heat flux perturbation
- $T(z,t)$: temperature perturbation from initial state $T_0$
- What is the temperature distribution at depth?

And more importantly(?), if we have so much wine that we feel compelled to store it underground, how deep a cellar do we need to prevent freezing?
Problem Setup:

- 1D homogenous half-space, surface at $z = 0$, thermal diffusion coefficient $\kappa$

- Governing PDE:
  \[
  \frac{\partial^2 T(z, t)}{\partial z^2} = \frac{1}{\kappa} \frac{\partial T(z, t)}{\partial t}.
  \]

- Boundary Conditions:
  \[
  T(0, t) = T_s(t),
  \quad T(\infty, t) = 0.
  \]
Fourier transform in time:

\[ \text{PDE} \rightarrow \text{ODE} \]

PDE:

\[
\frac{\partial^2 T(z, t)}{\partial z^2} = \frac{1}{\kappa} \frac{\partial T(z, t)}{\partial t}.
\]

Fourier transform in time:

\[
\int_{-\infty}^{\infty} \frac{\partial^2 T(z, t)}{\partial z^2}e^{-2\pi if t}dt = \int_{-\infty}^{\infty} \frac{1}{\kappa} \frac{\partial T(z, t)}{\partial t}e^{-2\pi if t}dt.
\]

Use derivative property:

\[ \text{FT}[\partial g/\partial t] = 2\pi if\text{FT}[g] \]

to obtain Fourier-domain ODE.

Let:

\[
\tilde{T}(z, f) = \text{FT}[T(z, t)] = \int_{-\infty}^{\infty} T(z, t)e^{-2\pi if t}dt
\]

Then:

\[
\frac{\partial^2}{\partial z^2} \tilde{T}(z, f) = \frac{2\pi if}{\kappa} \tilde{T}(z, f).
\]
Solve the ODE:
\[
\frac{\partial^2}{\partial z^2} \tilde{T}(z, f) = \frac{2\pi i f}{\kappa} \tilde{T}(z, f).
\]

• This second order ODE has a general solution:

\[
\tilde{T}(z, f) = c_1(f) \exp(+i\alpha z) + c_2(f) \exp(-i\alpha z)
\]

\[
\alpha = \left[ \frac{2\pi i f}{\kappa} \right]^{1/2}
\]

• Note:

\[
i^{1/2} = [e^{i\pi/2}]^{1/2} = e^{i\pi/4} = \frac{1 + i}{\sqrt{2}}.
\]

• Apply the boundary conditions:

\[
T(0, t) = T_s(t), \quad c_1(f) = \tilde{T}_s(f)
\]
\[
T(\infty, t) = 0, \quad c_2(f) = 0
\]

• Therefore:

\[
\tilde{T}(z, f) = \tilde{T}_s(f) \exp(-z/d_f) \exp(iz/d_f)
\]

Where \(d_f = \sqrt{\kappa / \pi f}\)
$d_f$: the attenuation ("skin") depth

$$d_f = \sqrt{\frac{\kappa}{\pi f}}$$

- Temperature perturbation decays with depth over this length-scale
- Increases with conductivity
- Decreases with frequency
  - Conductive time scale $\tau \sim d_f^2/\kappa$
- So we need deeper cellars to protect from steady (low-frequency) surface temperature perturbations
Heat flux boundary condition

• What if, instead of \( T(0,t) \), we know \( q(0,t) \)?

• Then, Fourier’s law gives us the boundary condition

\[
q_s(t) = k \frac{\partial T(z, t)}{\partial z}
\]

• From our earlier work, we know the solution will take the form:

\[
\tilde{T}(z, f) = c_3(f) \exp (+i\alpha z),
\]

• Recall from before that

\[
i\alpha = (i - 1)/d_f,
\]

• To find \( c_3(f) \) differentiate both sides w.r.t. \( z \):

\[
\frac{\partial}{\partial z} \tilde{T}(z, f) = c_3(f)i\alpha \exp (+i\alpha z) = c_3(f) \left( \frac{i - 1}{d_f} \right) \exp \left[ \frac{(i - 1)z}{d_f} \right]
\]
Use the BC to obtain the Fourier-domain solution:

• We know that:

\[
\frac{\partial}{\partial z} \tilde{T}(z, f) = c_3(f) i \alpha \exp(+i \alpha z) = c_3(f) \left( \frac{i - 1}{d_f} \right) \exp \left[ \frac{(i - 1)z}{d_f} \right]
\]

• Equate this expression at \( z = 0 \) with \( \tilde{q}_s(f)/\rho c_p \kappa \) to satisfy the surface heat boundary condition

• After a bit of algebra, we arrive at the solution

\[
\tilde{T}(z, f) = - \left( \frac{i + 1}{2 \rho c_p \sqrt{\pi \kappa f}} \right) \tilde{q}_s(f) \exp(-z/d_f) \exp(iz/d_f).
\]
Example: Sinusoidal surface temperature (e.g., daily cycle)

- Simple (but plausible) example with straightforward analytical solution
  - A model of temporal cycles with fixed frequency $f_0$
  - This example provides insight with analytical solution
  - In general, however, temperature distribution at depth is obtained numerically (via FFT and IFFT)

- Surface boundary condition:

\[
T(0, t) = T_s(t) = \Delta T \cos(2\pi f_0 t) = \frac{\Delta T}{2} (e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}).
\]
Analytical solution for $T(z,t)$

1. Fourier transform $T_s(t)$:

   $$
   \tilde{T}_s(f) = \int_{-\infty}^{\infty} \frac{\Delta T}{2} (e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}) e^{-2\pi if t} dt
   $$
   
   $$
   = \frac{\Delta T}{2} [\delta(f - f_0) + \delta(f + f_0)].
   $$

2. Plug into our formula for $T(z,f)$:

   $$
   \tilde{T}(z,f) = \tilde{T}_s(f) \exp(z(-1 + i)/d_f)
   $$
   
   $$
   = \frac{\Delta T}{2} [\delta(f - f_0) + \delta(f + f_0)] \exp(z(-1 + i)/d_f)
   $$
Analytical solution for $T(z,t)$

3. Invert the Fourier transform:

$$T(z,t) = \int_{-\infty}^{\infty} \frac{\Delta T}{2} [\delta(f - f_0) + \delta(f + f_0)] \exp(z(-1 + i)/d_f) e^{+2\pi i ft} df$$

$$= 2 \int_{0}^{\infty} \frac{\Delta T}{2} [\delta(f - f_0)] \exp(z(-1 + i)/d_f) e^{+2\pi i ft} df.$$ 

4. Time Domain Solution:

$$T(z,t) = \Delta T \exp\left(-z/\sqrt{\kappa/\pi f_0}\right) \cos\left(2\pi f_0 t - z/\sqrt{\kappa/\pi f_0}\right)$$

- Temperature perturbation attenuates with depth.
- Depth-dependent phase delay (proportional to conductive time scale).
Skin depths for temporal cycles on Earth (assume nominal $\kappa = 10^{-6} \text{ m}^2\text{s}^{-1}$):

$$d_f = \sqrt{\frac{\kappa}{\pi f}}$$

- **Daily cycle**
  - $f_0 = 1/\text{day}$: $d_f \sim 0.2 \text{ m}$

- **Seasonal cycle**
  - $f_0 = 1/\text{year}$: $d_f \sim 3 \text{ m}$

- **Glacial cycle**
  - $f_0 = 1/(10,000 \text{ years})$: $d_f \sim 300 \text{ m}$
Using a time series of the surface temperature of Mars, we can determine how the temperature anomaly will propagate to depth.

http://mars.jpl.nasa.gov/msl/images/PIA16105u_malin04MAINIMAGE-full.jpg
Air and ground temperature profiles on Mars were obtained by NASA’s Curiosity Rover. Data were taken by the Rover Environmental Monitoring Station between August 16 and August 17, 2012.

These plots show the temperature profiles at five depths (0 to 40 cm) over a period of just over 1 Martian day (≈1 day 40 min on Earth). Thermal diffusivity is varied for each plot (highest top left, lowest bottom left). Note that as depth increases, the temperature curve is smoothed, meaning the higher frequency components are preferentially quenched.

The temperature converges around -50 degrees Celsius. I guess we’ll have to look elsewhere for our wine cellar...
Conclusions

• Wine is not safe on Mars.
• Perturbations to surface temperature attenuate with depth, and experience a time lag proportional to the diffusive timescale.
• Attenuation is frequency-dependent: higher frequencies attenuate more quickly with depth.
• The propagation of surface temperature anomalies is highly dependent on the thermal diffusivity.
• Only a very thin layer, < 1m, is affected by the daily thermal surface fluctuations. Longer cycles, such as glacial periods, can affect hundreds of meters deep.