Heat Diffusion in Ice

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Photo credit: Sinead Farrell
Goals

- Examine how heat is redistributed within glacier ice as its boundary is heated or cooled
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\[ T(z, t) = T_0 + \Delta T e^{-z\sqrt{\frac{\omega}{2\kappa}}} \sin \left( \omega t - z\sqrt{\frac{\omega}{2\kappa}} \right) \]
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• Examine how heat is redistributed within glacier ice as its boundary is heated or cooled

\[ T(z,t) = T_0 + \Delta T e^{-\frac{z}{2\kappa}} \sin \left( \omega t - z\sqrt{\frac{\omega}{2\kappa}} \right) \]

• Plot the variation of temperature with depth
Assumptions

• Ice velocity is negligible
• Advection is negligible
• Ice is homogeneous
• Ice is isotropic
• Horizontal ice surface is heated uniformly
  o (heat flow occurs only in the vertical direction)
• Ice can be treated as infinite half-space
  o Temperature far from surface equals $T_0$ for all time $t$
  o (ice is really thick and influence of lower boundary on near-surface temperatures is not important)
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Assumptions

- Ice velocity is negligible
- Advection is negligible
- Ice is homogeneous
- Ice is isothermal
- Horizontal ice surface is heated uniformly
  - Temperature far from surface equals $T_0$ for all time $t$
    - Ice is really thick and influence of lower boundary on near-surface temperatures is not important

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• Top ~15m of a glacier subject to:
  o Diurnal variations (solar)
  o Annual variations (solar & atmospheric)
  o Longer variations (climate changes)
\[ T(z,t) = T_0 + Y_1(z)\cos(\omega t) + Y_2(z)\sin(\omega t) \]
$T(z, t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t)$

\[
\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}
\]
\[ T(z, t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

\[ \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} \]

\[ Y_1 = -\kappa \frac{\omega}{dy^2} \frac{d^2 Y_2}{dy^2} \]

\[ Y_2 = \kappa \frac{\omega}{dy^2} \frac{d^2 Y_1}{dy^2} \]
\[ T(z,t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

\[ \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} \]

\[ Y_1 = -\frac{\kappa}{\omega} \frac{d^2 Y_2}{dy^2} \]

\[ Y_2 = \frac{\kappa}{\omega} \frac{d^2 Y_1}{dy^2} \]

\[ \frac{d^4 Y_2}{dy^4} + \frac{\omega^2}{\kappa^2} Y_2 = 0 \]
\[ Y_2 = c_1 e^{\alpha \tau} + c_2 e^{-\alpha \tau} \]
Temperature must be finite when $z \rightarrow \infty$

$$Y_z = c_1 e^{\alpha z} + c_2 e^{-\alpha z}$$
\[ Y_2 = c_1 e^{\alpha z} + c_2 e^{-\alpha z} \]

\[ \alpha^4 + \frac{\omega^2}{\kappa^2} Y_2 = 0 \]
\[ Y_2 = c_1 e^{\alpha z} + c_2 e^{-\alpha z} \]

\[ \alpha^4 + \frac{\omega^2}{\kappa^2} Y_2 = 0 \]

\[ \alpha = -\left( \frac{1 \pm i}{\sqrt{2}} \right) \sqrt{\frac{\omega}{\kappa}} \]
\[ \alpha^4 + \frac{\omega^2}{\kappa^2} Y_2 = 0 \]

\[ Y_2 = c_1 e^{i\alpha} + c_2 e^{-i\alpha} \]

\[ \alpha = -\left( \frac{1 \pm i}{\sqrt{2}} \right) \sqrt{\frac{\omega}{\kappa}} \]

\[ Y_2 = b_1 e^{-\frac{i\alpha}{2 \kappa}} + b_2 e^{-\frac{i(1-\alpha)}{2 \kappa}} \]
\[
\alpha^4 + \frac{\omega^2}{\kappa} Y_2 = 0 \\
\alpha = -\frac{1 \pm i}{\sqrt{2}} \sqrt{\frac{\omega}{\kappa}}
\]

\[
Y_2 = b_1 e^{-\frac{1+i}{\sqrt{2}} \frac{\omega}{\kappa} z} + b_2 e^{-\frac{1-i}{\sqrt{2}} \frac{\omega}{\kappa} z}
\]

\[
Y_2 = e^{-\frac{\omega}{\sqrt{2 \kappa} z}} \left( b_1 e^{\frac{i \omega}{\sqrt{2 \kappa} z}} + b_2 e^{-i \frac{\omega}{\sqrt{2 \kappa} z}} \right)
\]
\[ Y_2 = c_1 e^{i \omega \kappa z} + c_2 e^{-i \omega \kappa z} \]

\[ \alpha^4 + \frac{\omega^2}{\kappa^2} Y_2 = 0 \]

\[ \alpha = -\left( \frac{1 \pm i}{\sqrt{2}} \right) \sqrt{\frac{\omega}{\kappa}} \]

\[ Y_2 = b_1 e^{-i\omega \sqrt{2k\kappa}} + b_2 e^{i\omega \sqrt{2k\kappa}} \]

\[ Y_2 = e^{-\sqrt{\frac{\omega}{2k}} z} \left( b_1 e^{i\sqrt{\frac{\omega}{2k}} z} + b_2 e^{-i\sqrt{\frac{\omega}{2k}} z} \right) \]

\[ Y_2 = e^{\sqrt{\frac{\omega}{2k}} z} \left( a_1 \cos \sqrt{\frac{\omega}{2k}} z + a_2 \sin \sqrt{\frac{\omega}{2k}} z \right) \]
\[ \alpha^4 + \frac{\omega^2}{K^2} Y_2 = 0 \]

\[ \alpha = -\left( \frac{1 \pm i}{\sqrt{2}} \right) \sqrt{\frac{\omega}{K}} \]

\[ Y_2 = b_1 e^{-(1+i) \sqrt{\frac{\omega}{2K}z}} + b_2 e^{-(1-i) \sqrt{\frac{\omega}{2K}z}} \]

\[ Y_2 = e^{-\sqrt{\frac{\omega}{2K}z}} \left( b_1 e^{i \sqrt{\frac{\omega}{2K}z}} + b_2 e^{-i \sqrt{\frac{\omega}{2K}z}} \right) \]

\[ Y_2 = e^{-\sqrt{\frac{\omega}{2K}z}} \left( a_1 \cos \sqrt{\frac{\omega}{2K}z} + a_2 \sin \sqrt{\frac{\omega}{2K}z} \right) \]

…repeat for \( Y_1 \)

\[ Y_1 = e^{-\sqrt{\frac{\omega}{2K}z}} \left( a_3 \cos \sqrt{\frac{\omega}{2K}z} + a_4 \sin \sqrt{\frac{\omega}{2K}z} \right) \]
\[ T(z,t) = T_0 + Y_1(z)\cos(\omega t) + Y_2(z)\sin(\omega t) \]

\[ Y_1 = e^{-\frac{\omega}{2\kappa}z} \left( a_3 \cos \sqrt{\frac{\omega}{2\kappa}z} + a_4 \sin \sqrt{\frac{\omega}{2\kappa}z} \right) \]

\[ Y_2 = e^{-\frac{\omega}{2\kappa}z} \left( a_1 \cos \sqrt{\frac{\omega}{2\kappa}z} + a_2 \sin \sqrt{\frac{\omega}{2\kappa}z} \right) \]
\[ T(z,t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

\[ Y_1 = e^{-\frac{\omega}{\sqrt{2K}} z} \left( a_3 \cos \left( \frac{\omega}{2K} z \right) + a_4 \sin \left( \frac{\omega}{2K} z \right) \right) \]

\[ Y_2 = e^{-\frac{\omega}{\sqrt{2K}} z} \left( a_1 \cos \left( \frac{\omega}{2K} z \right) + a_2 \sin \left( \frac{\omega}{2K} z \right) \right) \]

\[ Y_1 = -\frac{\kappa}{\omega} \frac{d^2 Y_2}{dy^2} \]

\[ Y_2 = \frac{\kappa}{\omega} \frac{d^2 Y_1}{dy^2} \]
\[ T(z,t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

\[
Y_1 = e^{-\frac{\omega}{\sqrt{2} \kappa} z} \left( a_3 \cos \frac{\omega}{2 \kappa} z + a_4 \sin \frac{\omega}{2 \kappa} z \right) \\
Y_2 = e^{-\frac{\omega}{\sqrt{2} \kappa} z} \left( a_1 \cos \frac{\omega}{2 \kappa} z + a_2 \sin \frac{\omega}{2 \kappa} z \right)
\]

\[
Y_1 = -\frac{\kappa}{\omega} \frac{d^2 Y_2}{dy^2} \\
Y_2 = \frac{\kappa}{\omega} \frac{d^2 Y_1}{dy^2}
\]

\[
\begin{align*}
Y_1 & = -\frac{\kappa}{\omega} \frac{d^2 Y_2}{dy^2} \\
Y_2 & = \frac{\kappa}{\omega} \frac{d^2 Y_1}{dy^2}
\end{align*}
\]

\[
\begin{align*}
a_2 & = a_3 \\
a_1 & = -a_4
\end{align*}
\]
\[ T(z,t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

\[ Y_1 = e^{-\frac{\omega}{\sqrt{2k}z}} \left( a_3 \cos \frac{\omega}{2k}z + a_4 \sin \frac{\omega}{2k}z \right) \]

\[ Y_2 = e^{-\frac{\omega}{\sqrt{2k}z}} \left( a_1 \cos \frac{\omega}{2k}z + a_2 \sin \frac{\omega}{2k}z \right) \]

\[ Y_1 = -\frac{\kappa}{\omega} \frac{d^2Y_2}{dy^2} \quad \text{a}_2 = a_3 \]

\[ Y_2 = \frac{\kappa}{\omega} \frac{d^2Y_1}{dy^2} \quad \text{a}_1 = -a_4 \]
\[ T(z,t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

\[ Y_1 = e^{-\frac{\omega}{\sqrt{2\kappa} z}} \left( a_3 \cos \sqrt{\frac{\omega}{2\kappa} z} + a_4 \sin \sqrt{\frac{\omega}{2\kappa} z} \right) \]

\[ Y_2 = e^{-\frac{\omega}{\sqrt{2\kappa} z}} \left( a_3 \cos \sqrt{\frac{\omega}{2\kappa} z} + a_4 \sin \sqrt{\frac{\omega}{2\kappa} z} \right) \]
\[ T(z,t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

\[ Y_1 = e^{-\frac{\sqrt{\omega}}{2\kappa} z} \left( a_3 \cos \sqrt{\frac{\omega}{2\kappa}} z + a_4 \sin \sqrt{\frac{\omega}{2\kappa}} z \right) \quad \quad Y_2 = e^{-\frac{\sqrt{\omega}}{2\kappa} z} \left( a_1 \cos \sqrt{\frac{\omega}{2\kappa}} z + a_2 \sin \sqrt{\frac{\omega}{2\kappa}} z \right) \]

\[ T(\infty,t) = T_0 \]

\[ T(0,t) = T_0 + \Delta T \sin(\omega t) \]
\[ T(z,t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

\[ Y_1 = e^{-\frac{\omega}{\sqrt{2\kappa}} z} \left( a_3 \cos \frac{\omega}{2\kappa} z + a_4 \sin \frac{\omega}{2\kappa} z \right) \]

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\[ a_2 = 0 \]

\[ T(0, t) = T_0 + \Delta T \sin(\omega t) \]
\[ T(z, t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

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Y_1 = e^{-\frac{\omega}{\sqrt{2\kappa} z}} \left( a_3 \cos \sqrt{\frac{\omega}{2\kappa} z} + a_4 \sin \sqrt{\frac{\omega}{2\kappa} z} \right)
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Y_2 = e^{-\frac{\omega}{\sqrt{2\kappa} z}} \left( a_1 \cos \sqrt{\frac{\omega}{2\kappa} z} + a_2 \sin \sqrt{\frac{\omega}{2\kappa} z} \right)
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T(\infty, t) = T_0
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T(0, t) = T_0 + \Delta T \sin(\omega t)
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\[
a_1 = \Delta T
\]

\[
a_2 = 0
\]
\[ Y_1 = e^{-\frac{\omega}{\sqrt{2\kappa}}} \left( -\Delta T \sin \frac{\omega}{2\kappa} z \right) \]

\[ Y_2 = e^{-\frac{\omega}{\sqrt{2\kappa}}} \left( \Delta T \cos \frac{\omega}{2\kappa} z \right) \]
\[ T(z, t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

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T(z,t) = T_0 + Y_1(z)\cos(\omega t) + Y_2(z)\sin(\omega t)
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Y_1 = e^{-\sqrt{\frac{\omega}{2k}}z} \left(-\Delta T \sin \sqrt{\frac{\omega}{2k}}z\right)
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\[
Y_2 = e^{-\sqrt{\frac{\omega}{2k}}z} \left(\Delta T \cos \sqrt{\frac{\omega}{2k}}z\right)
\]

\[
T(z,t) = T_0 + \Delta T e^{-\sqrt{\frac{\omega}{2k}}z} \left(\sin(\omega t) \cos \sqrt{\frac{\omega}{2k}}z - \cos(\omega t) \sin \sqrt{\frac{\omega}{2k}}z\right)
\]
\[ T(z,t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

\[ Y_1 = e^{-\sqrt{2/\kappa} \cdot z} \left( -\Delta T \sin \left( \frac{\omega}{2 \kappa} z \right) \right) \]

\[ Y_2 = e^{-\sqrt{2/\kappa} \cdot z} \left( \Delta T \cos \left( \frac{\omega}{2 \kappa} z \right) \right) \]

\[ T(z,t) = T_0 + \Delta T e^{-\sqrt{2/\kappa} \cdot z} \left( \sin(\omega t) \cos \left( \frac{\omega}{2 \kappa} z \right) - \cos(\omega t) \sin \left( \frac{\omega}{2 \kappa} z \right) \right) \]

\[ \sin \left( \omega t - z \sqrt{\frac{\omega}{2 \kappa}} \right) \]
\[ T(z, t) = T_0 + Y_1(z) \cos(\omega t) + Y_2(z) \sin(\omega t) \]

\[ Y_1 = e^{-\frac{\omega z}{2\kappa}} \left( -\Delta T \sin \sqrt{\frac{\omega}{2\kappa}} z \right) \]

\[ Y_2 = e^{-\frac{\omega z}{2\kappa}} \left( \Delta T \cos \sqrt{\frac{\omega}{2\kappa}} z \right) \]