Heat Flow Across a Fracture Zone
Group D

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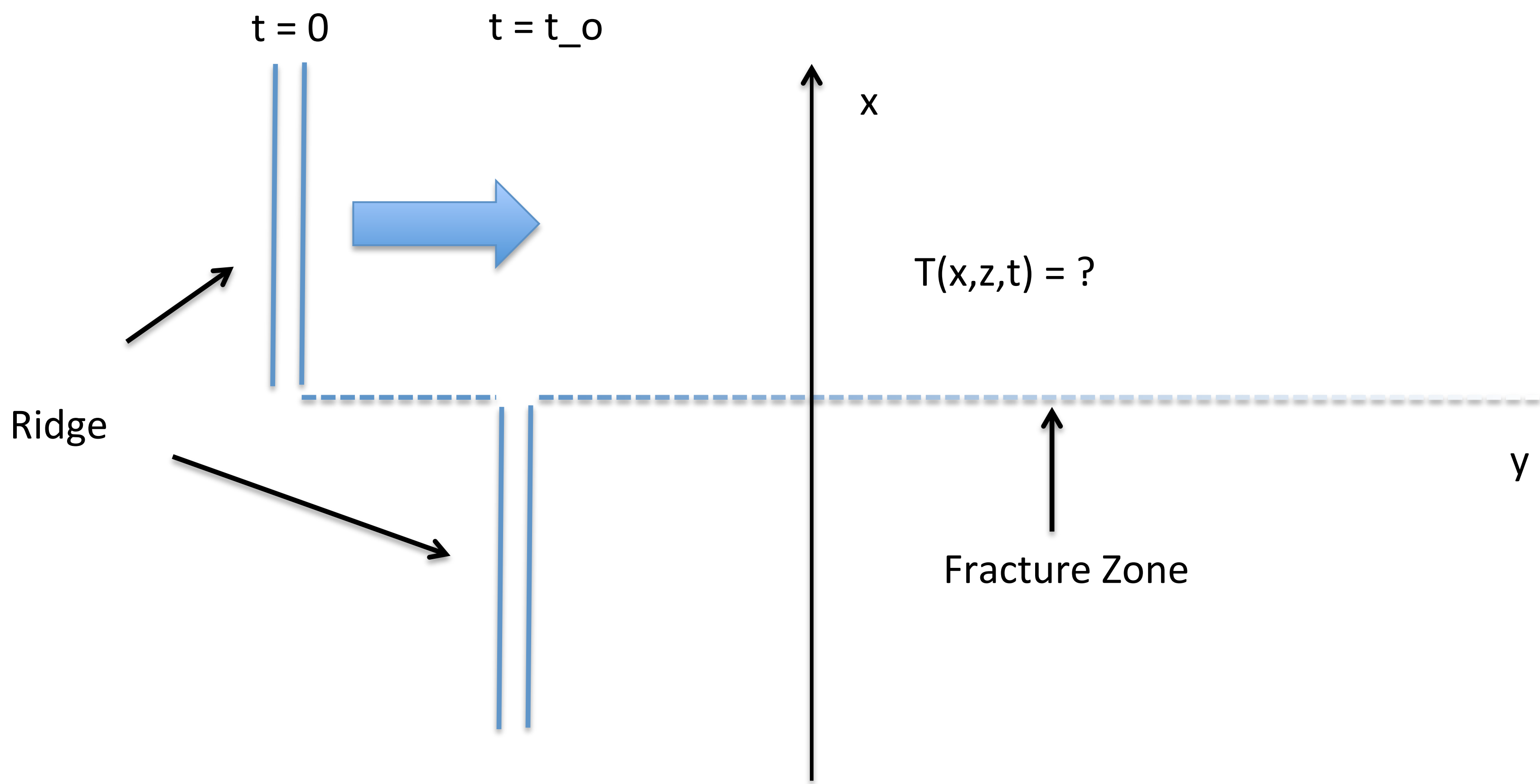
Can we find the temperature across a fracture zone?



$$\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}$$

$$T(x, z, t_o) = T_m \qquad (x < 0)$$

$$T(x, z, t_o) = T_m \operatorname{erf}\left(\frac{z}{2\sqrt{\kappa t_o}}\right) \qquad (x > 0)$$

$$T(x, 0, t) = 0$$

$$T(x, \infty, t) = T_m$$

$$\theta = \frac{T - T_o}{T_m - T_o}$$

$$\frac{\partial^2 \theta}{\partial z^2} + \frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\kappa} \frac{\partial \theta}{\partial t}$$

$$\theta(x, z, t_o) = 1 \qquad (x < 0)$$

$$\theta(x, z, t_o) = \text{erf}(\frac{z}{2\sqrt{\kappa t_o}}) \qquad (x > 0)$$

$$\theta(x, 0, t) = 0$$

$$\theta(x, \infty, t) = 1$$

$$\theta(x, z, t_o) = -1 \qquad (x < 0 \text{ and } z < 0)$$

$$\theta(x, z, t_o) = \text{erf}(\frac{z}{2\sqrt{\kappa t_o}}) \qquad (x > 0 \text{ and } z < 0)$$

How to solve a PDE using Green's functions:

1. Find your Green's function
   - It's the solution to your PDE without the BC's (sort of)

2. Find an analytic expression for the initial state of your system

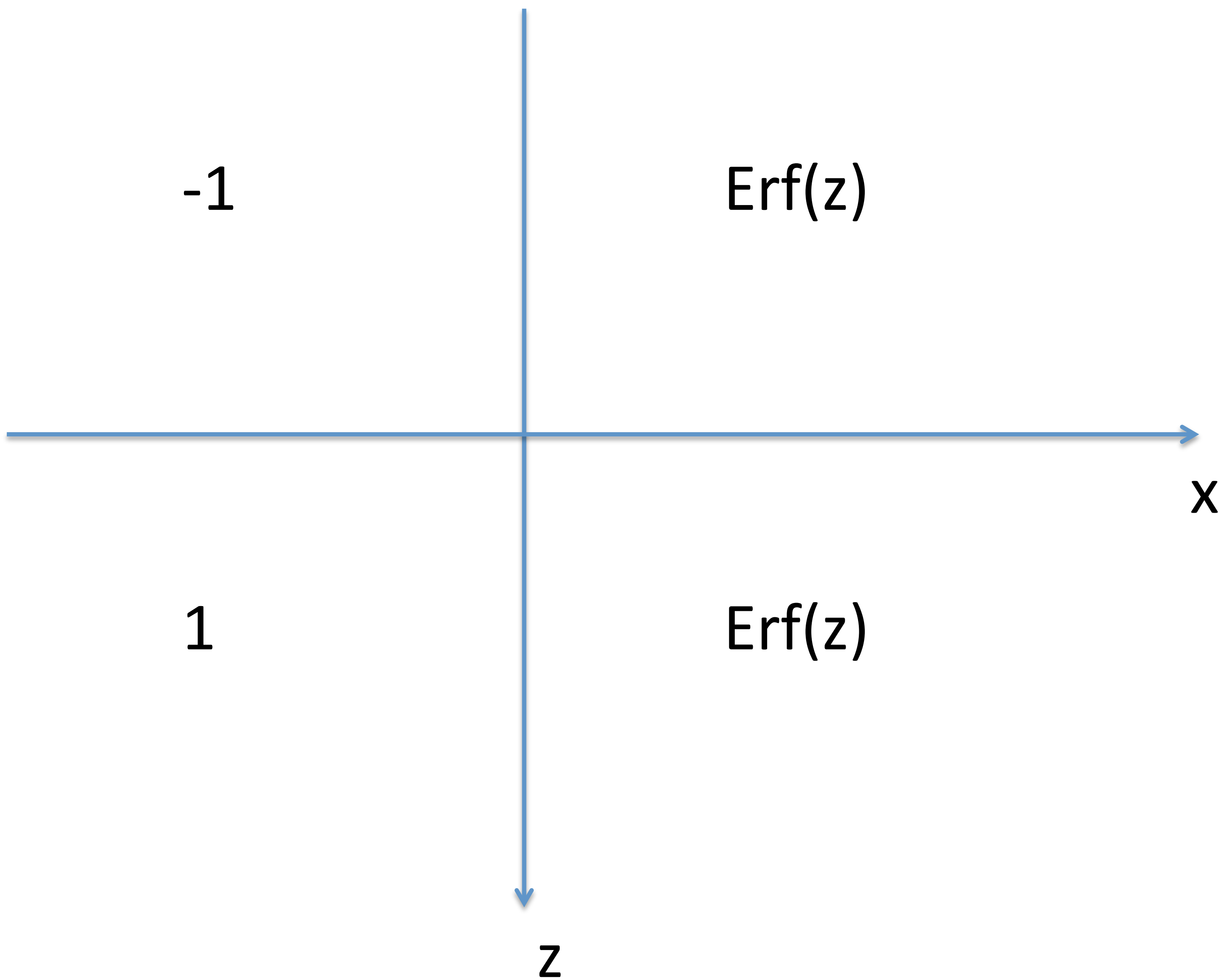
3. Convolve (1) with (2) to get the specific solution!

$$\frac{\partial^2\theta}{\partial z^2} + \frac{\partial^2\theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t}$$

$$(i2\pi k_x)^2\theta + (i2\pi k_z)^2\theta = \frac{1}{\kappa}\frac{\partial\theta}{\partial t}$$

$$-4\pi^2\kappa(k_x^2 + k_z^2)\theta = \frac{\partial\theta}{\partial t}$$

$$\tilde{\theta}(k_x, k_z, t) = G(k_x, k_z, t) = e^{-4\pi^2\kappa(k_x^2+k_z^2)t}$$


-1
Erf(z)
x
1
Erf(z)
z

$$f_o = (1 - H(x))(2H(z) - 1) + H(x)\mathrm{erf}(\frac{z}{2\sqrt{\kappa t}})$$

$$FT(H(x)) = \frac{1}{2}\delta(k_x) + \frac{1}{i2\pi k_x}$$

$$FT(\mathrm{erf}(\frac{z}{2\sqrt{\kappa t_o}})) = \frac{e^{-4\pi^2 \kappa t_o k_z^2}}{i\pi k_z}$$

$$F_o = (-\frac{1}{2}\delta(k_x) + \frac{1}{i2\pi k_x})\frac{1}{i\pi k_z} + (\frac{1}{2}\delta(k_x) + \frac{1}{i2\pi k_x})\frac{e^{-4\pi^2 \kappa t_o k_z^2}}{i\pi k_z}$$

$$F_oG = \frac{1}{i\pi k_z}\left[\left(-\frac{1}{2}\delta(k_x)+\frac{1}{i2\pi k_x}\right)+\left(\frac{1}{2}\delta(k_x)+\frac{1}{i2\pi k_x}\right)e^{-4\pi^2\kappa t_o k_z^2}\right]e^{-4\pi^2\kappa k_z^2 t}e^{-4\pi^2\kappa k_x^2 t}$$

$$FT^{-1}\left(\frac{e^{-4\pi^2\kappa k_z^2 t}}{i\pi k_z}\left(-\frac{1}{2}\delta(k_x)+\frac{1}{i2\pi k_x}\right)e^{-4\pi^2\kappa k_x^2 t}\right) = \text{erf}\left(\frac{z}{2\sqrt{\kappa(t-t_o)}}\right)\left(-\frac{1}{2}\delta(k_x)+\frac{1}{i2\pi k_x}\right)e^{-4\pi^2\kappa k_x^2 t}$$

$$FT^{-1}\left(\text{erf}\left(\frac{z}{2\sqrt{\kappa(t-t_o)}}\right)\left(-\frac{1}{2}\delta(k_x)+\frac{1}{i2\pi k_x}\right)e^{-4\pi^2\kappa k_x^2 t}\right) = \text{erf}\left(\frac{z}{2\sqrt{\kappa(t-t_o)}}\right)\frac{1}{2}\left(\text{erf}\left(\frac{x}{2\sqrt{\kappa(t-t_o)}}\right)-1\right.$$

$$\frac{1}{2}\left(\text{erf}\left(\frac{x}{2\sqrt{\kappa(t-t_o)}}\right)+1\right)\text{erf}\left(\frac{z}{2\sqrt{\kappa(t-t_o+t_o)}}\right)$$

$$= \frac{1}{2}\text{erfc}\left(\frac{-x}{2\sqrt{\kappa(t-t_o)}}\right)\text{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\theta(x,z,t) = \frac{1}{2}\text{erf}\left(\frac{z}{2\sqrt{\kappa(t-t_o)}}\right)\left(1-\text{erf}\left(\frac{x}{2\sqrt{\kappa(t-t_o)}}\right)\right)+\frac{1}{2}\left(1+\text{erf}\left(\frac{x}{2\sqrt{\kappa(t-t_o)}}\right)\right)\text{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$T(x,z,t) = \frac{T_m-T_o}{2}\left[\text{erf}\left(\frac{z}{2\sqrt{\kappa(t-t_o)}}\right)\left(1-\text{erf}\left(\frac{x}{2\sqrt{\kappa(t-t_o)}}\right)\right)+\left(1+\text{erf}\left(\frac{x}{2\sqrt{\kappa(t-t_o)}}\right)\right)\text{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right)\right.$$

And now, for something completely different…

Isostasy!

$$g\int_0^{z*}\rho_m \mathrm{dz} = g\int_0^d \rho_w \mathrm{dz} + g\int_d^{z*}\rho_m(1-\alpha(T-T_m))\mathrm{dz}$$

$$\int_0^d(\rho_w-\rho_m)\mathrm{dz} = \int_d^{z*}\rho_m\alpha(T-T_m)\mathrm{dz}$$

Now we just need to plug in for T…

Let's do it for a simple case first…

$$T(z,t) = (T_m - T_o)\mathrm{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right) + T_o$$

$$d(t) = \rho_m \alpha \frac{(T_m - T_o)}{(\rho_m - \rho_w)} \int_d^\infty \left(1 - \mathrm{erf}\left(\frac{z-d}{2\sqrt{\kappa t}}\right)\right) \mathrm{dz}$$

$$z' = z - d$$

$$d(t) = \rho_m \alpha \frac{(T_m - T_o)}{(\rho_m - \rho_w)} \int_0^\infty \mathrm{erfc}\left(\frac{z'}{2\sqrt{\kappa t}}\right) \mathrm{dz'}$$

$$\eta = \frac{z}{2\sqrt{\kappa t}}$$

$$\Rightarrow dz = 2\sqrt{\kappa t} d\eta$$

$$d(t) = \rho_m \alpha \frac{(T_m - T_o)}{(\rho_m - \rho_w)} 2\sqrt{\kappa t} \int_0^\infty \operatorname{erfc}(\eta) d\eta$$

$$\int_0^\infty \operatorname{erfc}(\eta) d\eta = \frac{1}{\sqrt{\pi}}$$

$$d(t) = 2\rho_m \alpha \frac{(T_m - T_o)}{(\rho_m - \rho_w)} \sqrt{\frac{\kappa t}{\pi}} + d_o$$

$$T(x,z,t) = \frac{T_m - T_o}{2}\left[\operatorname{erf}\left(\frac{z}{2\sqrt{\kappa(t-t_o)}}\right)\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{\kappa(t-t_o)}}\right)\right)\right.$$

$$\left. + \left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{\kappa(t-t_o)}}\right)\right)\operatorname{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right)\right]$$

$$T(x,z,t) = \frac{T_m - T_o}{2}\gamma_1 \operatorname{erf}(\frac{z}{2\sqrt{\kappa(t-t_o)}})) + \frac{T_o}{2}$$

$$\frac{T_m - T_o}{2}\gamma_2 \operatorname{erf}(\frac{z}{2\sqrt{\kappa t}})) + \frac{T_o}{2}$$

$$\gamma_1(x) = 1 - \operatorname{erf}(\frac{x}{2\sqrt{\kappa(t-t_o)}})$$

$$\gamma_2(x) = 1 + \operatorname{erf}(\frac{x}{2\sqrt{\kappa(t)}})$$

$$d(x,t) = \rho_m \alpha \frac{(T_m - T_o)}{(\rho_m - \rho_w)} \left(\sqrt{\frac{\kappa(t - t_o)}{\pi}} \gamma_1(x) + \sqrt{\frac{\kappa t}{\pi}} \gamma_2(x) \right) + d_o$$

… and that's it!

And now for some pretty pictures!

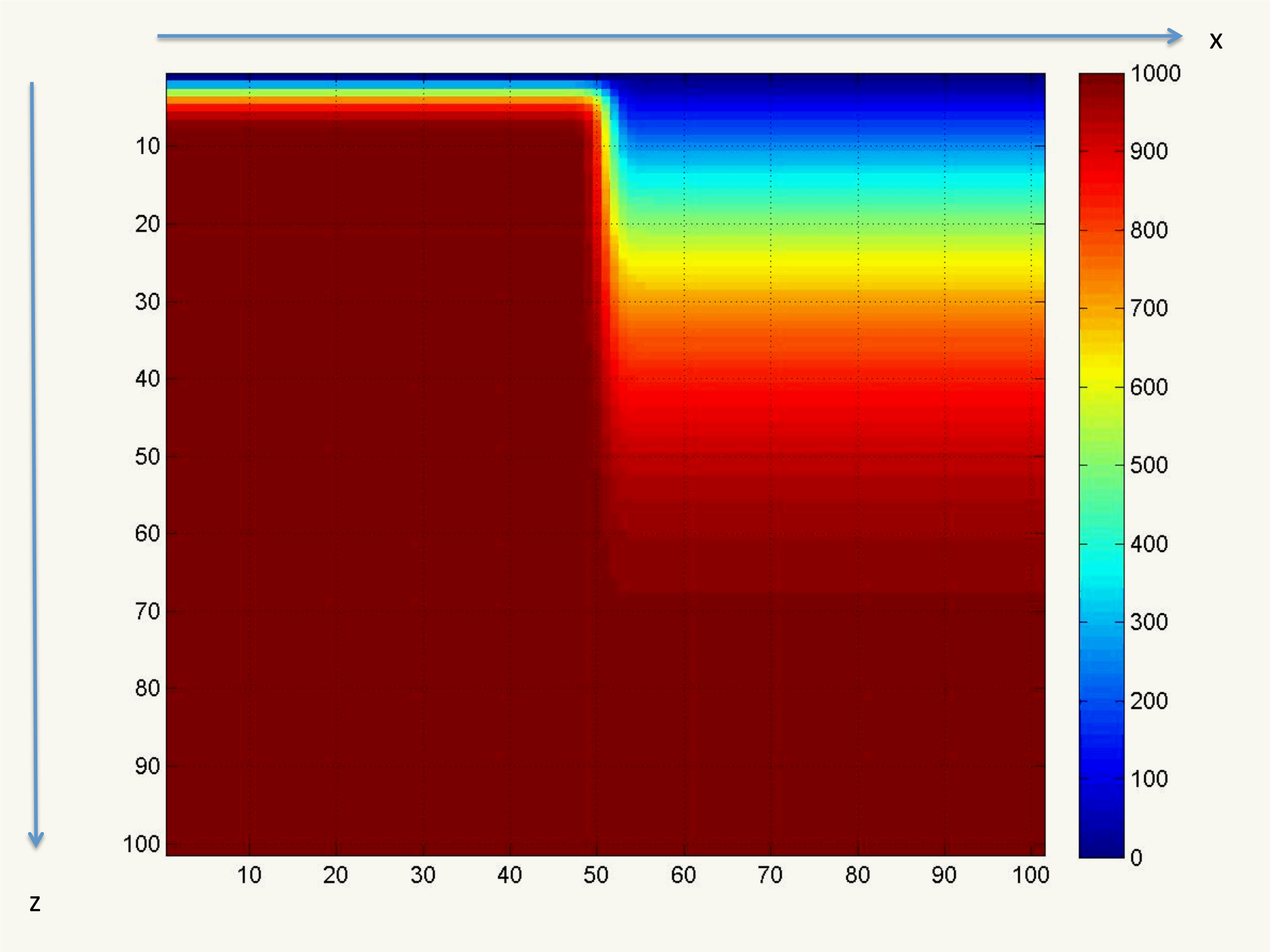
set parameters

$Tm = 1000℃,\quad \mathrm{T}_0 = 0℃$

$\kappa = 1\times10^{-6}\ \mathrm{m}^2/s$

$\mathrm{z}:\ \text{from 0km to 10km}$

$\mathrm{x}:\ \text{from -10km to 10 km}$


x
z
10
20
30
40
50
60
70
80
90
100
1000
900
800
700
600
500
400
300
200
100
0

set parameters

$$\alpha = 3\times10^{-5}C^{-1}$$

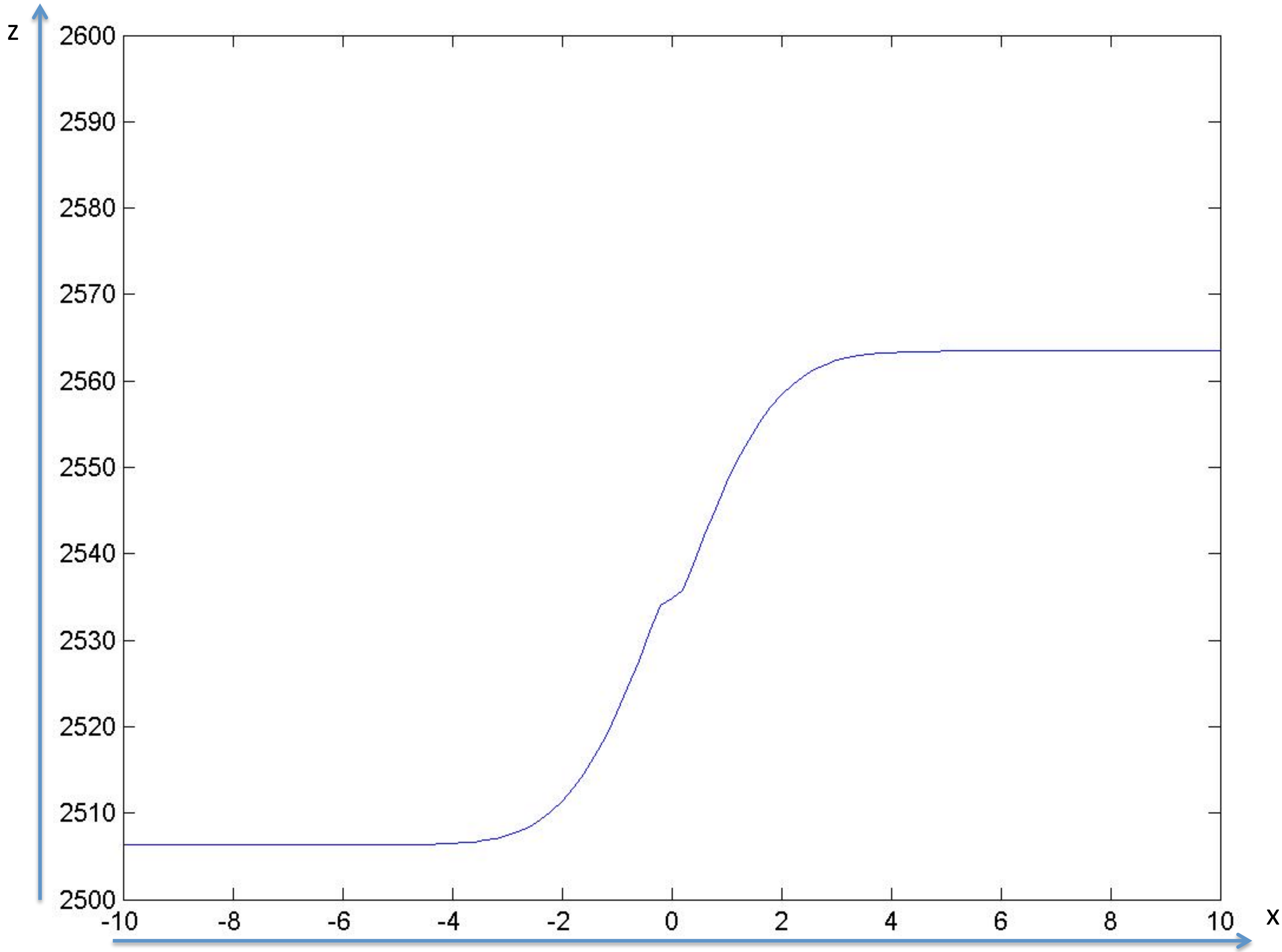
$$Tm = 1300℃，\ T_0 = 0℃$$

$$\kappa = 1\times10^{-6}\ \mathrm{m}^2/s$$

$$\mathrm{z:\ from\ 0km\ to\ 10km}$$

$$\mathrm{x:\ from\ -10km\ to\ 10\ km}$$

$$\mathrm{d}_0 = 2.5km$$


z
2600
2590
2580
2570
2560
2550
2540
2530
2520
2510
2500
-10
-8
-6
-4
-2
0
2
4
6
8
10
x