Heat flow from a plume

Brians House and Oller

Geodynamics

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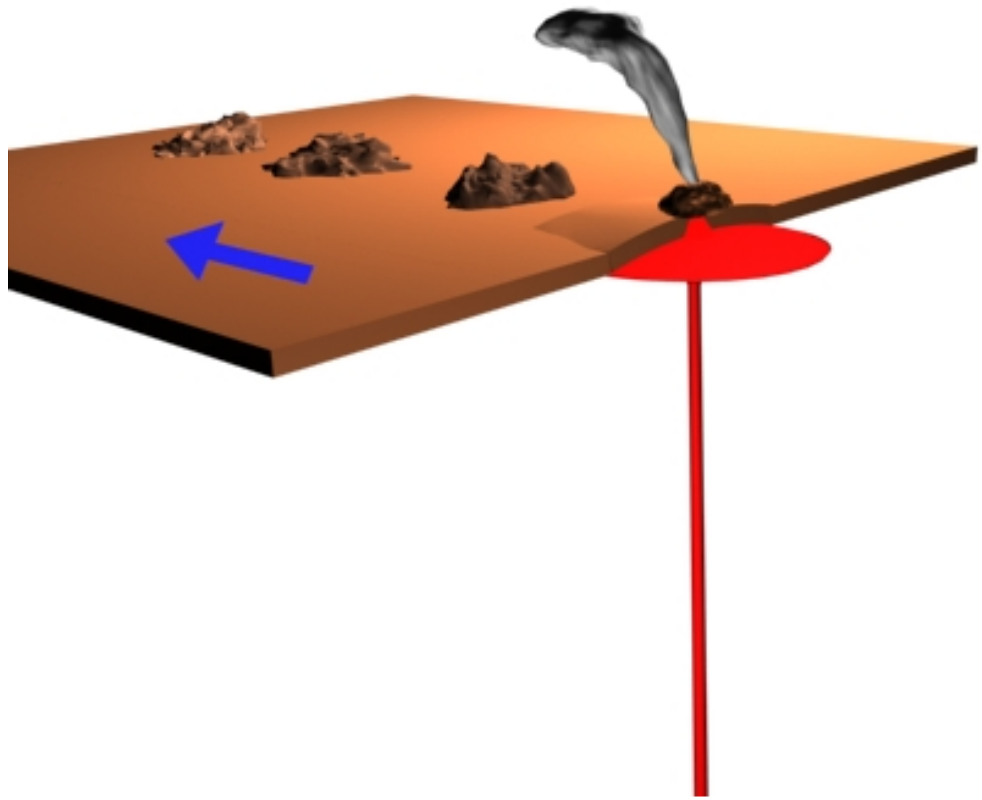
Mantle Plumes

- ✧ High temp material within the mantle
- ✧ Fixed position with respect to plate motion
- ✧ Reheats and thins lithosphere
- ✧ **Surface Expression:**
  - ✧ Volcanism
  - ✧ Large igneous provinces
  - ✧ Lithospheric swell



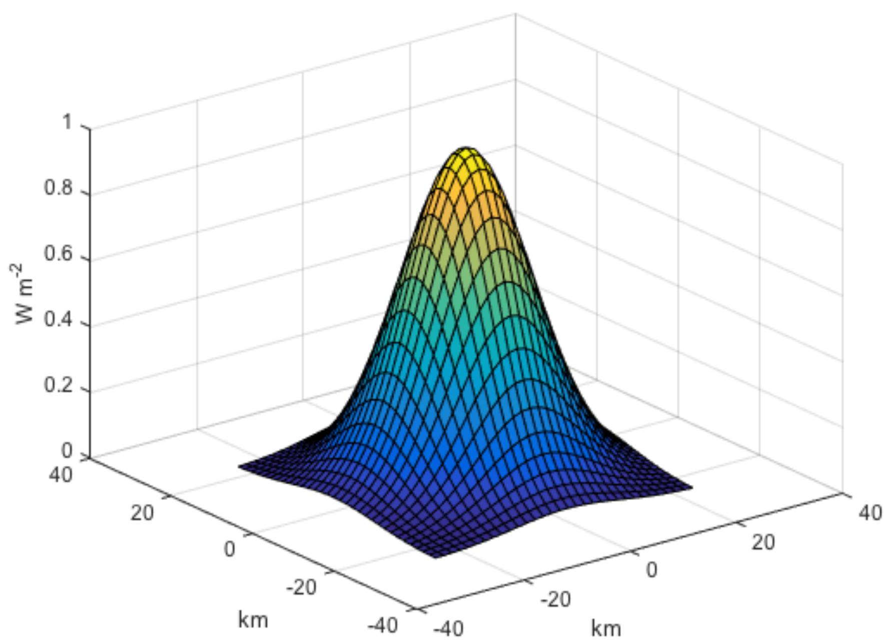
Governing PDE:

$$v \cdot \nabla T(x,y,z) - \kappa \nabla^2 T(x,y,z) = \frac{Q(x,y,z)}{c_p \rho_m}$$

advective heat transport - diffusive transport = heat production

Suppose Gaussian heat source:

$$Q(x,y,z) = A\delta(z - z_0)\exp\left[-\frac{x^2 + y^2}{2\sigma^2}\right]$$



With our usual Fourier Transform procedure:

$$-2\pi i(\mathbf{k}\cdot\mathbf{v})T(\mathbf{k}) - (-2\pi i)^2(\mathbf{k}\cdot\mathbf{k})\kappa T(\mathbf{k}) = \frac{Q(\mathbf{k})}{\rho_m c_p}$$

$$4\pi^2\left[(\mathbf{k}\cdot\mathbf{k}) - \frac{i(\mathbf{k}\cdot\mathbf{v})}{2\pi}\right]T(\mathbf{k}) = \frac{Q(\mathbf{k})}{\rho_m c_p \kappa}$$

Separating into horizontal, vertical components:

Let:

$$\mathbf{k}_h = k_x + k_y$$

$$Q_h = Ae^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$$

$$Q_z = \delta(z - z_0)$$

Then:

$$\mathcal{F}[Q_z](k_z) = \mathcal{F}[\delta(z - z_0)](k_z) = e^{-2\pi i k_z z_0}$$

This means:

$$4\pi^2\left[k_z^2 + (\mathbf{k}_h\cdot\mathbf{k}_h) - \frac{i(\mathbf{k}_h\cdot\mathbf{v})}{2\pi}\right]T(\mathbf{k}_h,k_z) = \frac{Q_v(k_z)Q_h(\mathbf{k}_h)}{\rho_m c_p \kappa} = \frac{e^{-2\pi i k_z z_0}Q_h(\mathbf{k}_h)}{\rho_m c_p \kappa}$$

Fancy time!

Let:

$$p^2 = (\mathbf{k}_h\cdot\mathbf{k}_h) - \frac{i(\mathbf{k}_h\cdot\mathbf{v})}{2\pi}$$

(isn't this obvious??)

$$\left(p^2 + k_z^2\right)T(\mathbf{k}_h,k_z) = \frac{Q_h(\mathbf{k}_h)}{4\pi^2\rho_m c_p \kappa}e^{-2\pi i k_z z_0}$$

So:

$$\left(p^2 + k_z^2\right) T(\mathbf{k}_h, k_z) = \frac{Q_h(\mathbf{k}_h)}{4\pi^2 \rho_m c_p \kappa} e^{-2\pi i k_z z_0}$$

$$T(\mathbf{k}_h, k_z) = \frac{Q_h(\mathbf{k}_h)}{4\pi^2 \rho_m c_p \kappa} \frac{e^{-2\pi i k_z z_0}}{\left(k_z - ip\right)\left(k_z + ip\right)}$$

We get:

$$T(\mathbf{k}_h, z) = \frac{Q_h(\mathbf{k}_h)}{4\pi^2 \rho_m c_p \kappa} \int_{-\infty}^{\infty} \frac{e^{-2\pi i k_z z_0} e^{2\pi i k_z z}}{\left(k_z - ip\right)\left(k_z + ip\right)} dk_z$$

Being Polish:

$$T(\mathbf{k}_h, z) = \frac{Q_h(\mathbf{k}_h)}{4\pi^2 \rho_m c_p \kappa} (2\pi i) \frac{e^{-2\pi i k_z z_0} e^{2\pi i k_z z}}{k_z + ip} \Big|_{k_z = ip}$$

Using the Cauchy Residue Theorem:

$$T(\mathbf{k}_h, z) = \frac{Q_h(\mathbf{k}_h)}{4\pi^2 \rho_m c_p \kappa} (2\pi i) \left(\frac{1}{2ip} \right) e^{-2\pi i(ip)z_0} e^{2\pi i(ip)z}$$

And finally to simplify:

$$T(\mathbf{k}_h, z) = \frac{Q_h(\mathbf{k}_h)}{4\pi \rho_m c_p \kappa\, p} e^{2\pi p(z_0 - z)}$$

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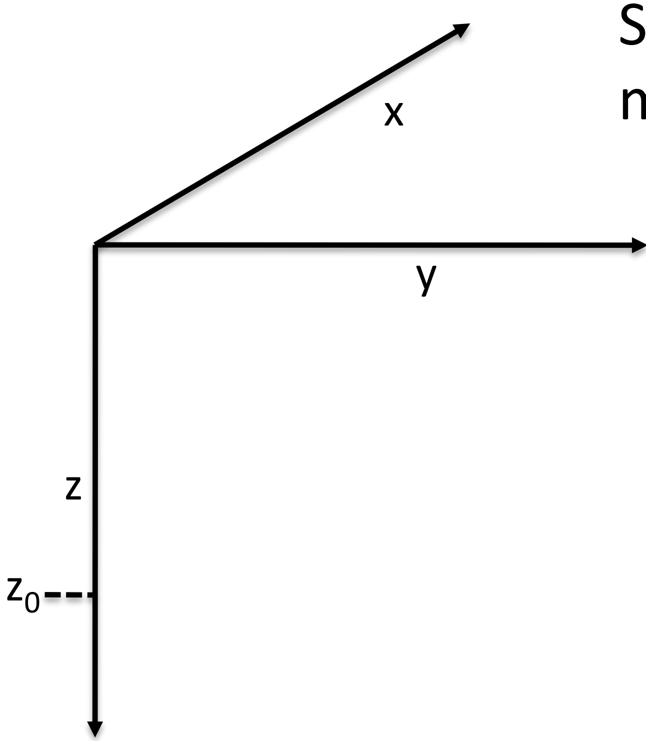
wait...



So $z > z_0$ for all z values in our model:

$$2\pi p(z_0 - z) < 0$$

And

$$e^{2\pi p(z_0 - z)}$$

will decay going up from z_0

Whew!

Or we could make it easier on ourselves…

$$T(\mathbf{k}_h, z) = \frac{Q_h(\mathbf{k}_h)}{4\pi\rho_m c_p \kappa\, p} e^{-2\pi p|z_0 - z|}$$

But will the fish be happy?

If we have:

$$T(\mathbf{k}_h, z) = \frac{Q_h(\mathbf{k}_h)}{4\pi\rho_m c_p \kappa p} e^{-2\pi p |z_0 - z|}$$

$$\Rightarrow T(\mathbf{k}_h, 0) = \frac{Q_h(\mathbf{k}_h)}{4\pi\rho_m c_p \kappa p} e^{-2\pi p |z_0|} \neq 0$$

Uhh oh…

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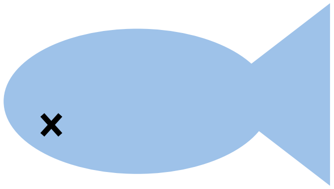

Final answer

Let's add an image (a heat sink) at $z = -z_0$ to force T(x,y,0) to 0

$$T(\mathbf{k}_h, z) = \frac{Q_h(\mathbf{k}_h)}{4\pi\rho_m c_p \kappa p}\left(e^{-2\pi p|z_0 - z|} - \underbrace{e^{-2\pi p(z_0+z)}}\right)$$

Heat sink term

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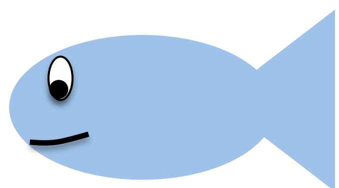
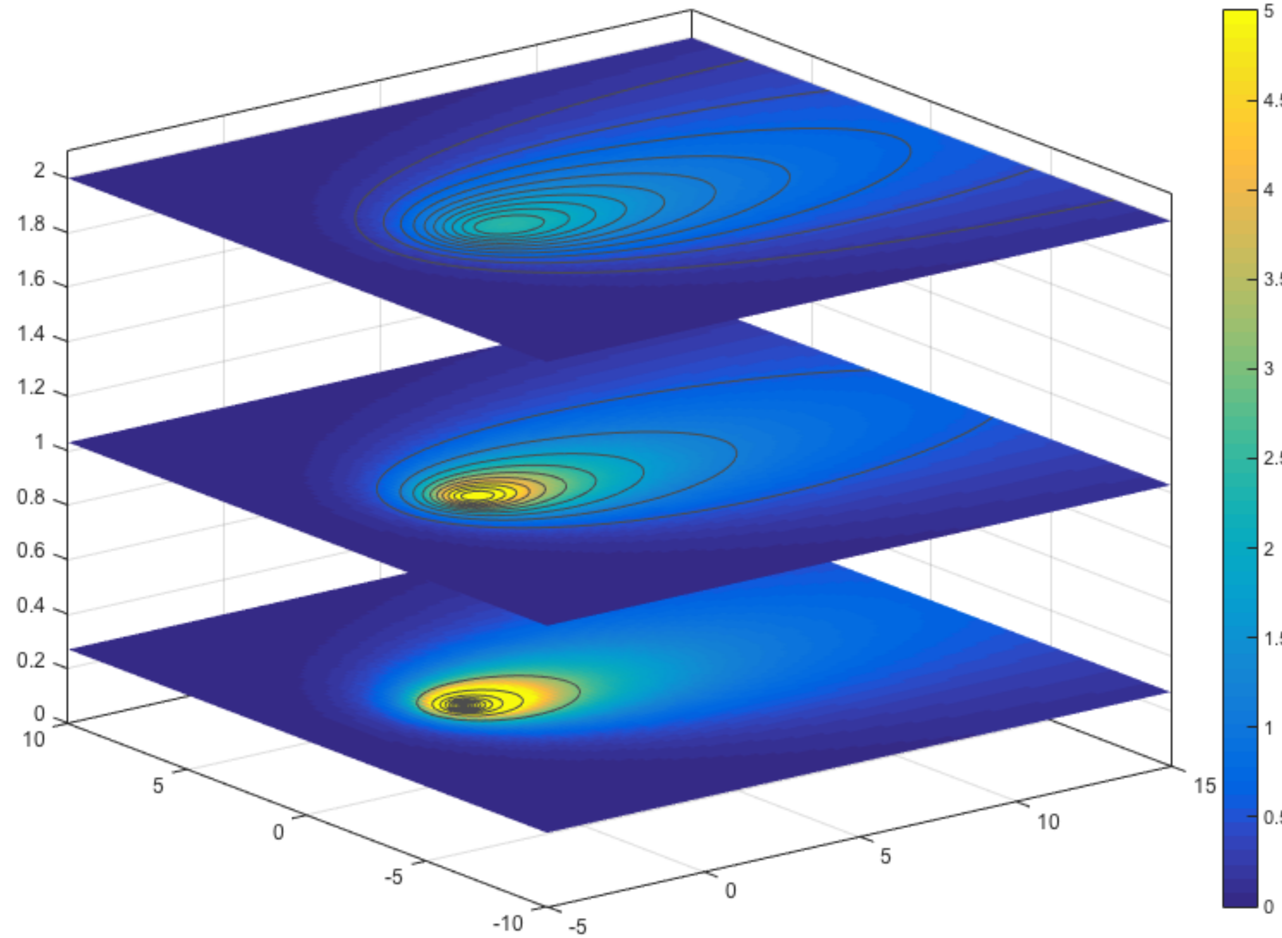


But what about pictures?



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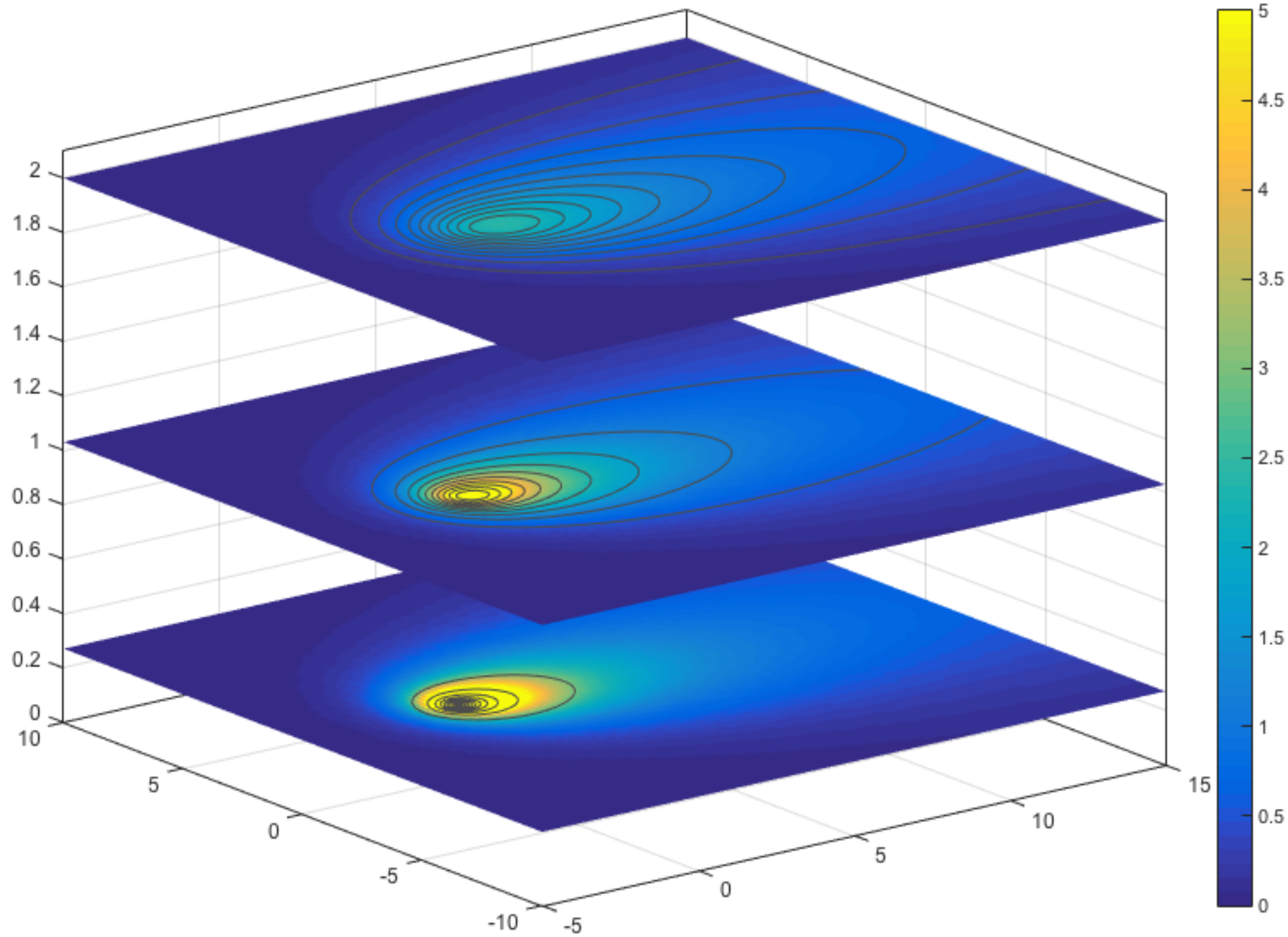
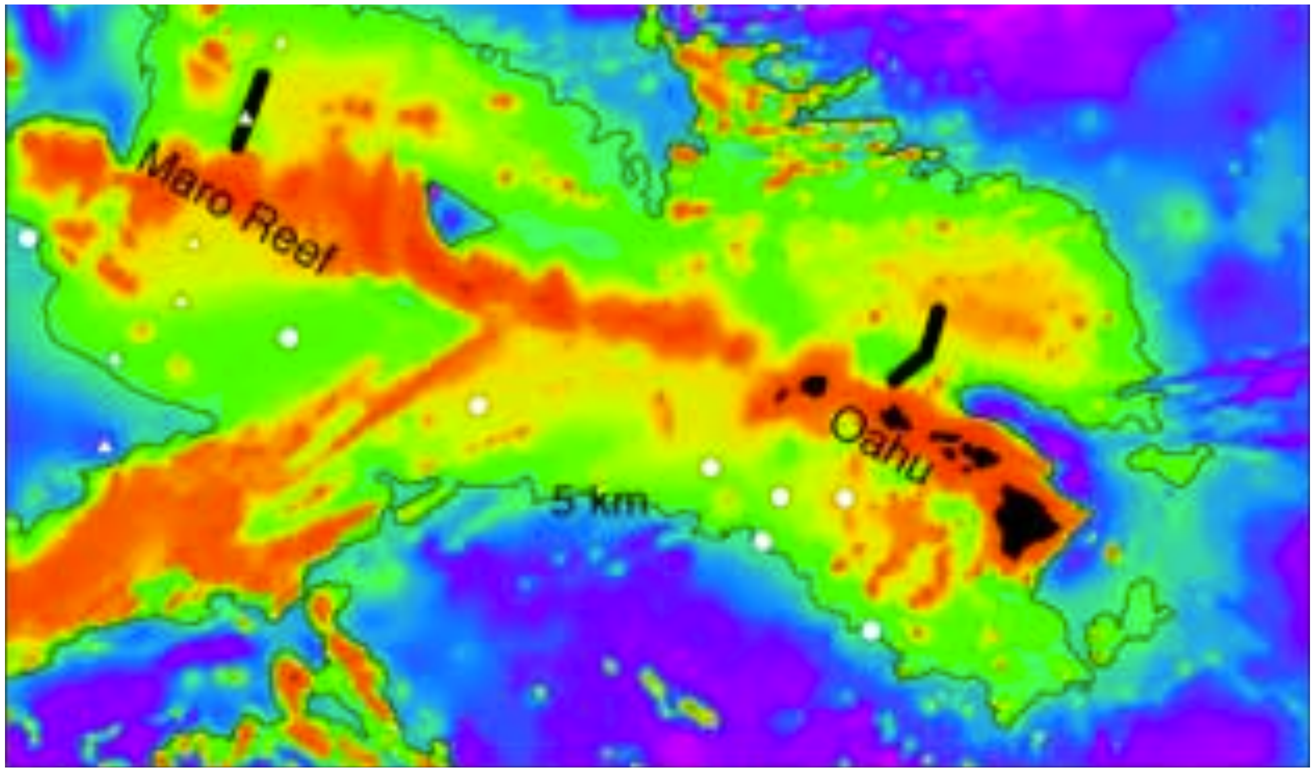


Units, you ask?

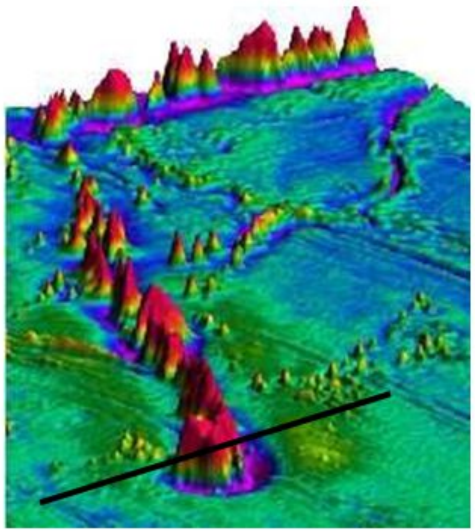
- The model works best when dimensionless
  - Otherwise $e^{-(\text{huge number})}$?
- Geologically sensible values still gave us trouble…

Figure on preceding slide is actually for Green's Function with (point) source at (x,y,z) = (0,0,0)

Observations

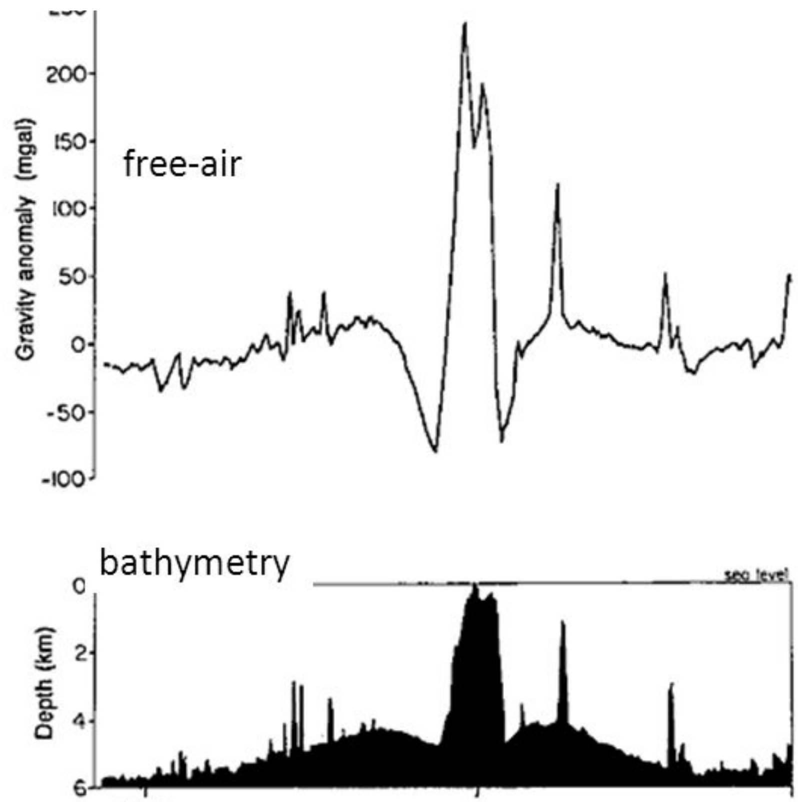


Gravity and Bathymetry



Two effects:

- Elastic flexure due to island load.
- A swell due to mantle upwelling.



How does the model compare?

So to summarize:

$$T(\mathbf{k}_h, z) = \frac{Q_h(\mathbf{k}_h)}{4\pi\rho_m c_p \kappa p}\left(e^{-2\pi p|z_0 - z|} - e^{-2\pi p(z_0 + z)}\right)$$

Inverse transform in x,y would give T(x,y,z) with T(x,y,0) = 0

- We didn't get a sensible heat model (!) through this procedure
- Perhaps better to use Green's Function to build source
  - Analytic approach using FT's may not be as practical as a numerical model

Oh, and this is a pretty complex problem…

References

- Sandwell, D., 1982. Thermal isostasy: Response of a moving lithosphere to a distributed heat source. *Journal of Geophysical Research*, 82, 1001-1014.

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- Robert Peterson, Dave Sandwell, personal communications.