The wine cellar problem

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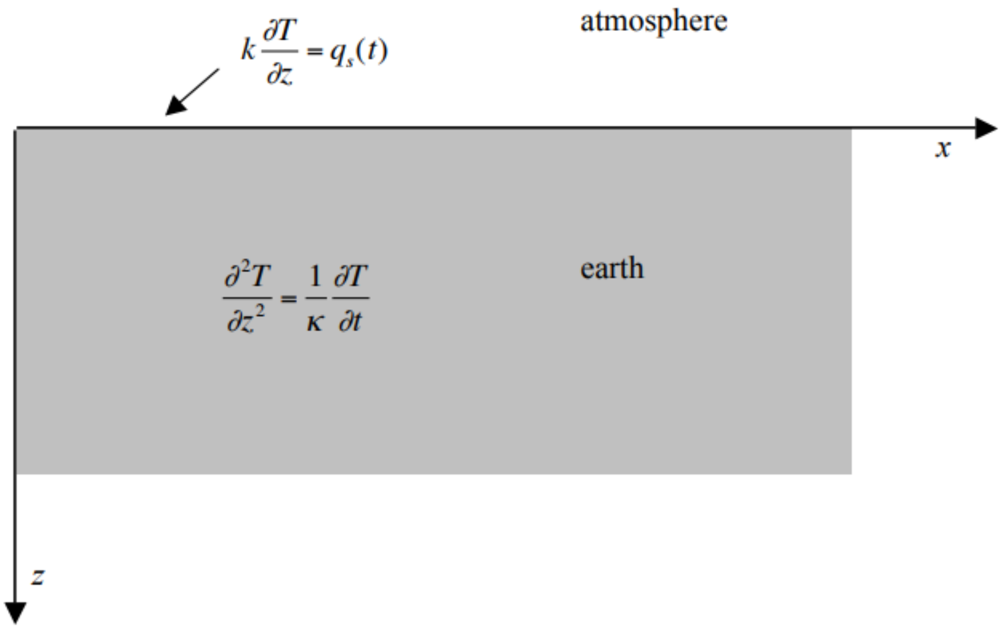
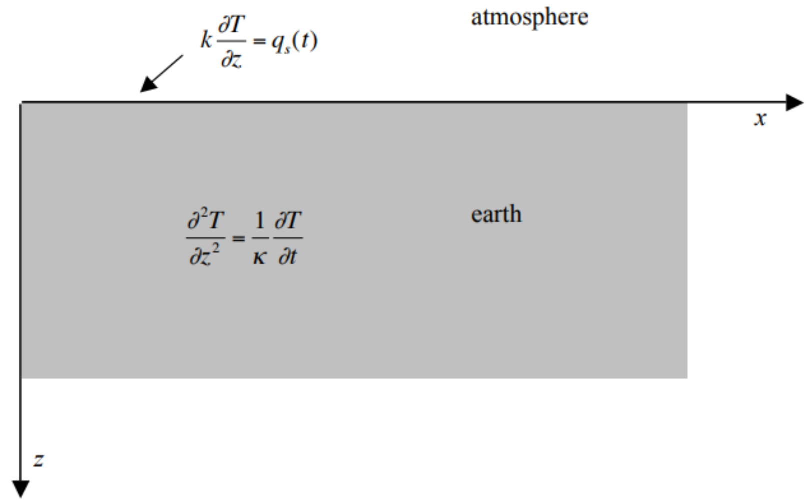
The temperature anomaly T at any time and depth $T(z,t)$



Assumptions:
- semi-infinite homogeneous half-space
- constant thermal properties.
- No heat production
- We assume a solution to q and T

First we assume that the solution to qs(t) is on the form:

$$q_s(t) = k\frac{\partial T}{\partial z} = Ae^{-i\omega t}$$



If we let $A^0 = \frac{A}{k}$ then, $\frac{\partial T}{\partial z} = A^0 e^{-i\omega t}$

And

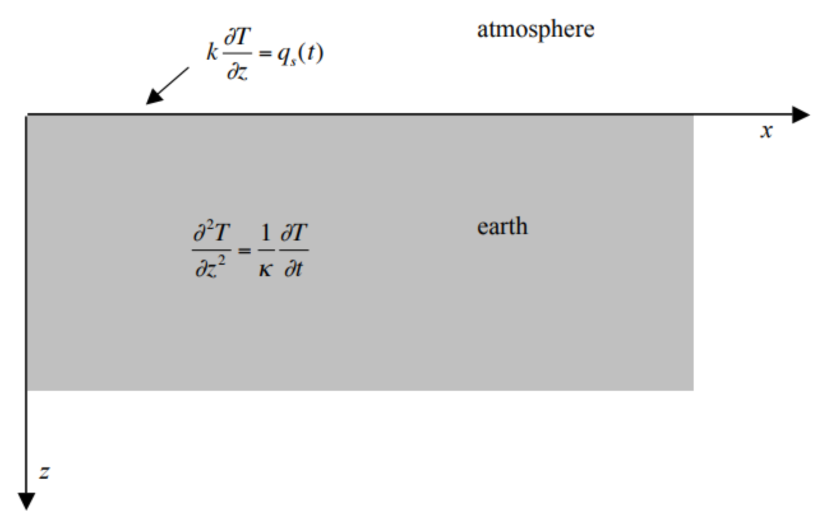
$$\boldsymbol{T(z,t) = A^0 f(z) e^{-i\omega t} + T_0}$$

Thus, we must find $A^0, f(z)$

$$k\frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}$$

Insert for T

$$T(z,t) = A^0 f(z)e^{-i\omega t} + T_0$$



Derivate with respect to t

$$A^0 f(z)e^{-i\omega t}\frac{\partial^2}{\partial z^2} = \boxed{\frac{1}{\kappa}A^0 f(z)e^{-i\omega t}\,dt}$$

$$\boxed{A^0 f(z)e^{-i\omega t}\frac{\partial^2}{\partial z^2}} = \frac{-i\omega}{\kappa}A^0 f(z)e^{-i\omega t}$$

This is now an ODE which we can solve by separating the variables and this way find our f(z)

Find f(z)

$$A^0 f(z) e^{-i\omega t} \frac{\partial^2}{\partial z^2}$$

remember ODE – separation of variables:
$Y'(x) = -\boldsymbol{\lambda\alpha} Y(x), Y''(x) = -\boldsymbol{\lambda} Y(x),$
$f(z) = ae^{\sqrt{-\lambda z}} + be^{-\sqrt{-\lambda z}}$
which in our case gives

$$\frac{-i\omega}{\kappa} \cancel{A^0} \cancel{f(z)} \cancel{e^{-i\omega t}} = -\lambda \cancel{A^0} \cancel{f(z)} \cancel{e^{-i\omega t}}$$

$$\lambda = \frac{i\omega}{\kappa}$$

a=0

$$f(z) = \cancel{ae^{\sqrt{-\lambda z}}} + be^{-\sqrt{-\lambda z}}$$

$$f(z) = e^{-\sqrt{-\frac{i\omega}{\kappa}}z}$$

Now we have f(z), we insert it in our equation:

$$T(z,t) = A^0 e^{-\sqrt{-\frac{i\omega}{\kappa}}z} e^{-i\omega t} + T_0$$

And finally, we need to find our new A^0

$$q_s(t) = k\frac{\partial T}{\partial z} = Ae^{-i\omega t}$$

$$Ae^{-i\omega t} = kA^0 e^{-\sqrt{-\frac{i\omega}{\kappa}}z} e^{-i\omega t} \frac{\partial}{\partial z}$$

$$A^0 = \frac{A}{k}\sqrt{\frac{\kappa}{\omega}} e^{i\frac{\pi}{4}}$$

pi/4 just happens to be an eighth of a cycle, meaning that if the sun hits its peak in the sky at noon, the warmest point in the day will be at 3 pm!

$$T(z,t) = \overbrace{\frac{A}{k}\sqrt{\frac{\kappa}{\omega}}e^{i\frac{\pi}{4}}}^{\text{f(z)}}\overbrace{e^{-\sqrt{-\frac{i\omega}{\kappa}}z}}^{A^0}e^{-i\omega t} \; + T_0$$

Remember that:

$$\sqrt{-i} = \frac{1-i}{\sqrt{2}}$$

$$T(z,t) = \frac{A}{k}\sqrt{\frac{\kappa}{\omega}}e^{i\frac{\pi}{4}}e^{-\frac{1-i}{\sqrt{2}}\sqrt{\frac{\omega}{\kappa}}z}e^{-i\omega t} \; + T_0$$

$$T(z,t) = \frac{A}{k}\sqrt{\frac{\kappa}{\omega}}e^{-\sqrt{\frac{\omega}{2\kappa}}z}e^{i(\frac{\omega}{2\kappa}z-\omega t+\frac{\pi}{4})} \; + T_0$$

Why did we just do this?

Case study: Oslo, Norway

We can now figure out where to put our wine cellar (or beer cave if that suits you better).

Or (more conveniently as grapes doesn’t grow in Norway) where to put our potato cellar.

First lets look at the attenuation of depth for the temperature anomaly in typical Norwegian bed rock


GRANITE
WET SOIL
SANDY SOIL

The attenuation depth

$\boldsymbol{\omega}$=2$\boldsymbol{\pi}$f, where f is found for the diurnal, annual and glacial cycle
$\boldsymbol{K}$ (thermal diffusivity) and k (thermal conductivity) are chosen for typical Norwegian bedrock

$$\boldsymbol{K} = \frac{k}{\boldsymbol{\rho} c} \qquad z_0 = \sqrt{\frac{2\boldsymbol{K}}{\boldsymbol{\omega}}}$$

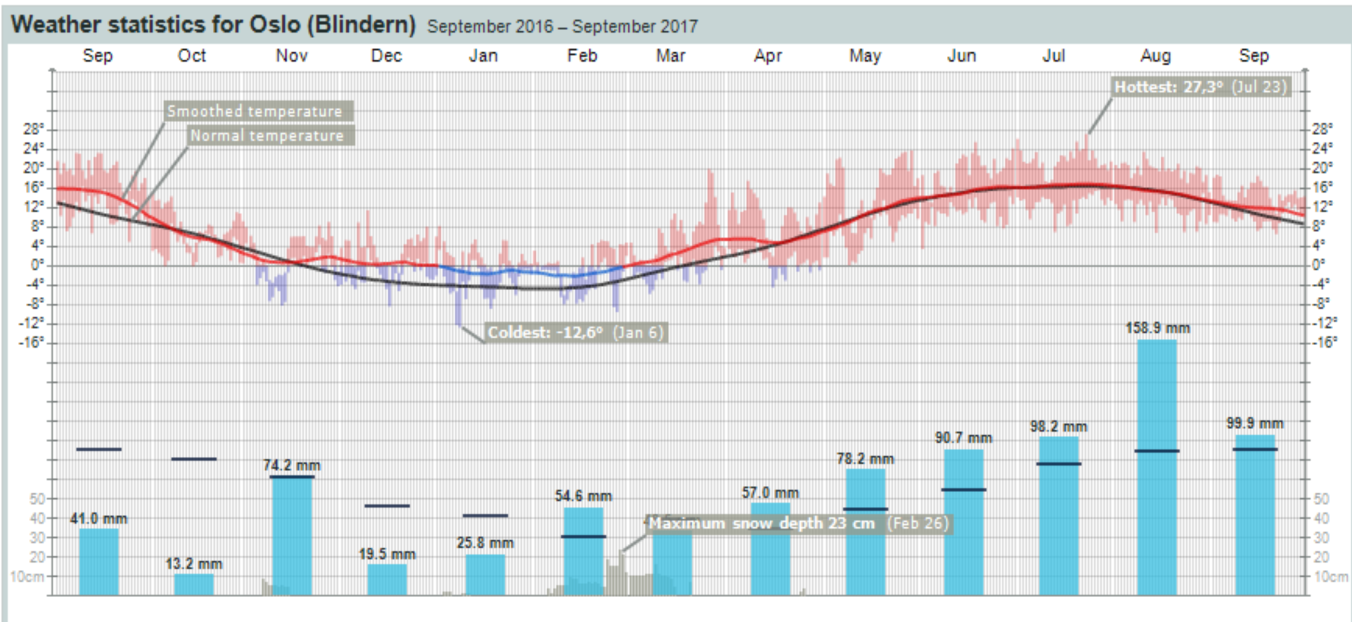
Thus only $\boldsymbol{K}$ and ω determine the attenuation depth of the temperature anomaly

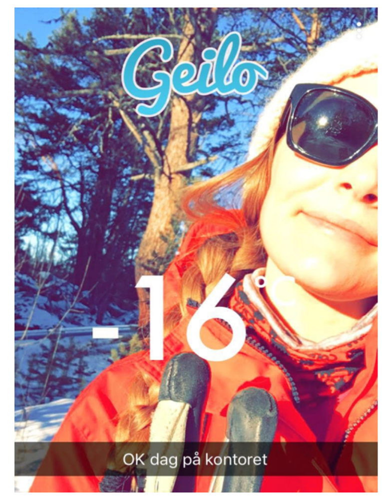
	Granite k= 3.62 J/kgC $\boldsymbol{K}$ = 1.67E-6 m^2/s	**Wet soil** k= 0.9 J/kgC $\boldsymbol{K}$ = 3.04E-7 m^2/s	**Sandy soil** K= 2 J/kgC $\boldsymbol{K}$ = 5.79E-7 m^2/s
Diurnal	0.21 m	0.09 m	0.13 m
Annual	4.09 m	1.75 m	2.41 m
Glacial	817 m	349 m	482 m

How does the temperature vary with depth in Oslo, Norway?
Where do we put our cellar?

$$T(z,t) = \frac{A}{k}\sqrt{\frac{\kappa}{\omega}}\, e^{-\sqrt{\frac{\omega}{2\kappa}}z}\, e^{i(\frac{\omega}{2\kappa}z - \omega t + \frac{\pi}{4})} + T_0$$

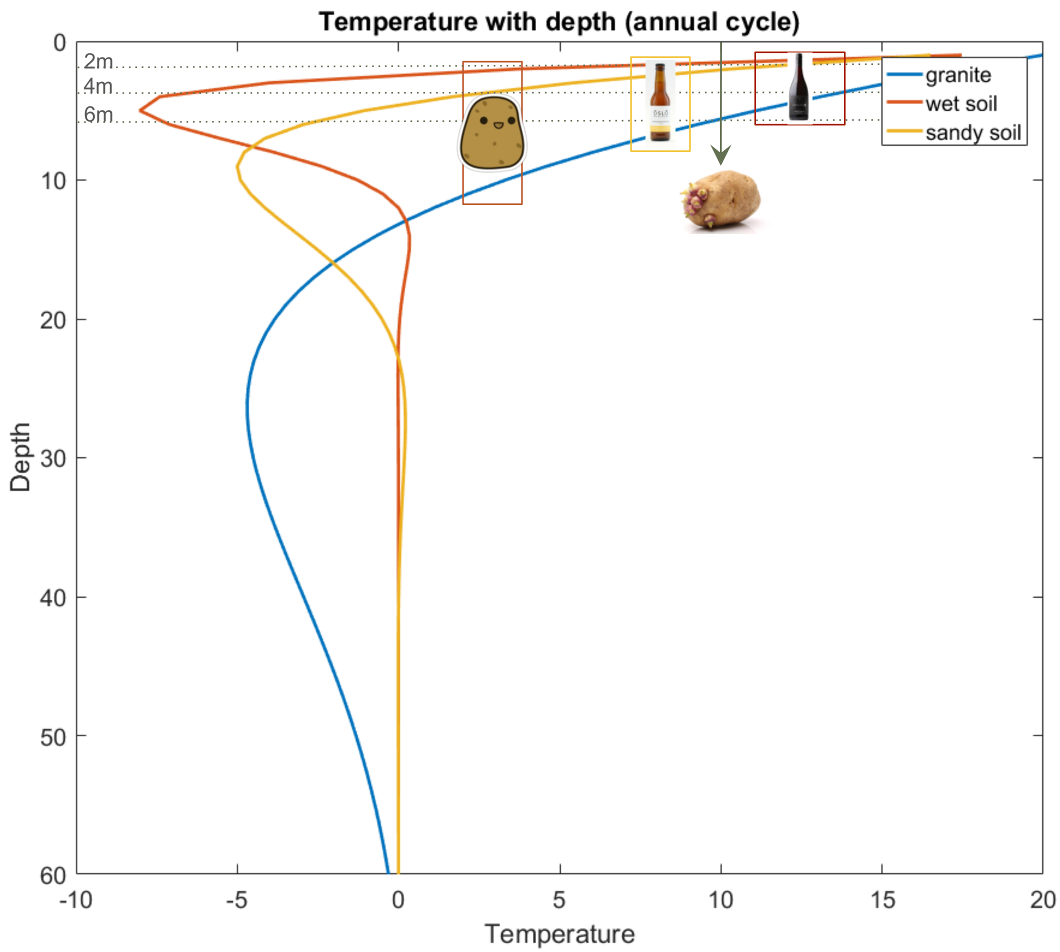
$\boldsymbol{\omega}_{annual} = 1.99 * 10^7$ rad/sec





$$A_{annual} = 39{,}9^{\circ}\, C$$
$$T_0 = 0^{\circ}\, C$$


KOEHRING


Temperature with depth (annual cycle)
2m
4m
6m
granite
wet soil
sandy soil
Depth
0
10
20
30
40
50
60
-10
-5
0
5
10
15
20
Temperature