The Gaussian Plume

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Our **volumetric heat production rate** per unit time looks like

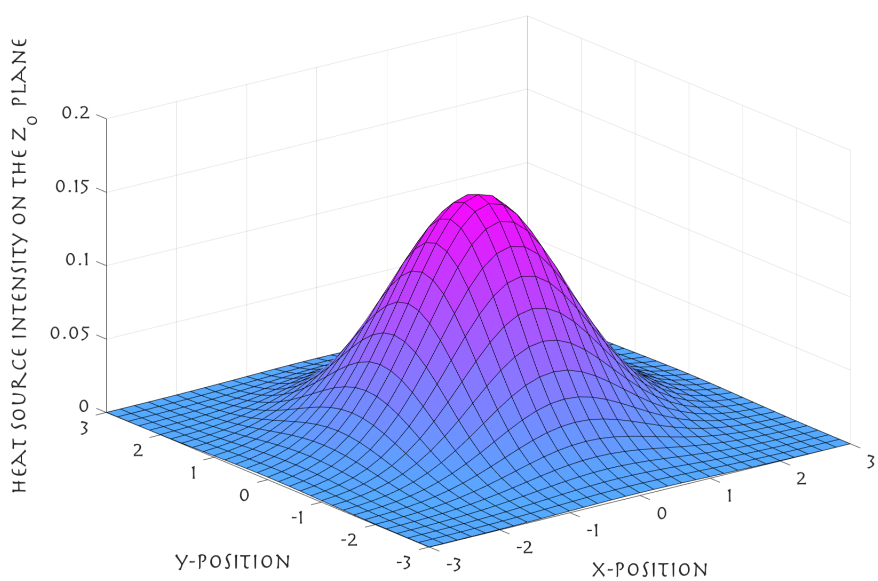

Figure 1: 2-D intensity plot.

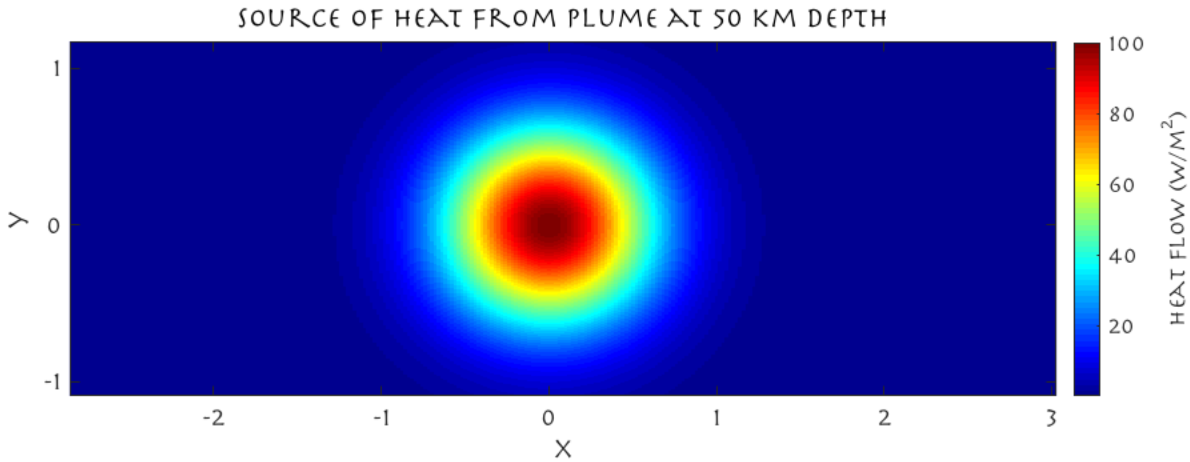

Figure 2: Aerial contour plot looking down at the $z = z_o$ plane.

and satisfies the equation

$$Q(x, y, z) = A\delta(z - z_0)e^{-\frac{x^2+y^2}{2\sigma^2}}. \tag{1}$$

1 The Differential Equation

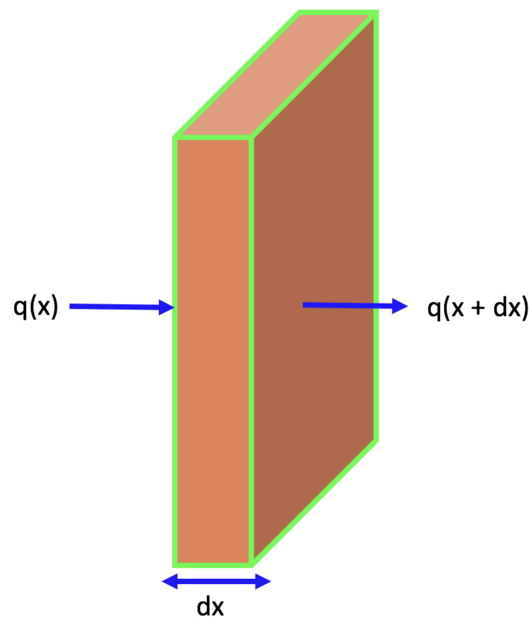

Figure 3

If we consider the heat flux (heat flow per unit area per unit time) through the slab in only one direction as $q(x)$, then

$$q(x + \partial x) - q(x) \approx q(x) + \frac{dq}{dx}\partial x - q(x) \tag{2}$$

$$= \frac{dq}{dx}\partial x = -k_x \frac{d^T}{dx^2}\partial x \tag{3}$$

is the net heat flux in or out of the slab. The net heat flux is logically decomposable into two, distinct components.

There is the component of outward heat flux that comes from heat sources within the slab. In math language, this is

$$q_{\text{source}} = Q(x)\partial x \tag{4}$$

If $Q(x)$ is the volumetric heat production rate. Any additional heat flux beyond this will draw from the internal energy within the slab and thus reduce the temperature with heat flux contribution

$$Q_{\Delta T} = -\rho C \frac{dT}{dt} \partial x \tag{5}$$

where C is the specific heat of the slab material and the negative sign indicates that a positive "amount" of heat escaping the slab will cause a **reduction** in the temperature. When we put these three things together, we get

$$-k_x \frac{d^2T}{dx^2} \partial x = Q(x)\partial x - \rho C \frac{dT}{dt} \partial x \tag{6}$$

$$\implies \frac{dT}{dt} - \kappa_x \frac{d^2T}{dx^2} = \frac{Q(x)}{\rho C}. \tag{7}$$

Then we consider the motion of the slab.

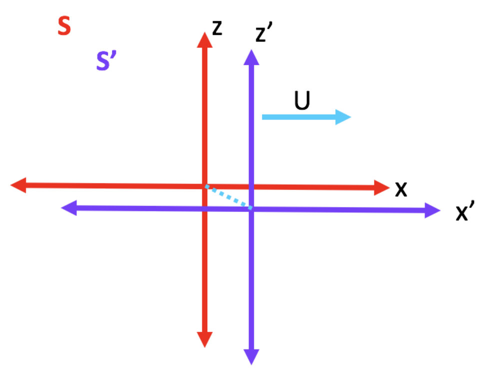

Figure 4

Everything done so far comes from looking at the slab from a reference frame in which the slab is not moving. If the slab *is* moving, the equation above works in the frame comoving with the slab. To obtain the equation describing the system from frame S instead of S', we notice that

$$x = x' + Ut \tag{8}$$

relates the position in space as seen by an observer in S to that seen by an observer in S'. Then the derivative of the temperature with respect to time as seen by an observer in S must be

$$\frac{dT}{dt} = \frac{dT}{dx}\frac{dx}{dt} = U\frac{dT}{dx} \tag{9}$$

while the derivative of temperature with respect to space does not change in form because

$$dx = dx'. \tag{10}$$

Thus, the equation for temperature behavior in the slab as seen from frame S is

$$U\frac{dT}{dx} - \kappa_x\frac{d^2T}{dx^2} = \frac{Q(x)}{\rho C}. \tag{11}$$

Extending this to 3-D, we finally have

$$\mathbf{v}\cdot\nabla T - \kappa\nabla^2 T = \frac{Q(x,y,z)}{\rho C} \tag{12}$$

2 Fourier Transform Method

$$\mathscr{F}\left(v_x\frac{dT}{dx} - \kappa\Big(\frac{d^2T}{dx^2} + \frac{d^2T}{dx^2} + \frac{d^2T}{dx^2}\Big) = A_o\delta(z-z_0)e^{-\frac{x^2+y^2}{2\sigma^2}}\right) \tag{13}$$

$$\left(2\pi i k_x v_x + 4\kappa\pi^2(k_x^2+k_y^2+k_z^2)\right)\tilde{T}(k_x,k_y,k_z) = A_o e^{-2\pi i k_z z_0}\mathscr{F}\left(e^{-\frac{x^2+y^2}{2\sigma^2}}\right) \tag{14}$$

where

$$\mathscr{F}\left(e^{-\frac{x^2+y^2}{2\sigma^2}}\right) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}e^{-2\pi i k_x x}dx\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}e^{-2\pi i k_y y}dy \tag{15}$$

$$= R(k_x)R(k_y) \quad\Big|\quad R(k) = \mathscr{F}(e^{-\frac{u^2}{2\sigma^2}}) \tag{16}$$

$$= \mathscr{F}(e^{-\pi\left(\frac{u}{\sigma\sqrt{2\pi}}\right)^2}) \tag{17}$$

$$= \mathscr{F}\Big(f(\alpha u)\Big) \tag{18}$$

where $f(u) = e^{-\pi u^2}$ and $\alpha = \frac{1}{\sigma\sqrt{2\pi}}$. From the **Scaling Property** of the Fourier transform,

$$= \frac{1}{|\alpha|}\tilde{f}(\frac{k}{\alpha}) \tag{19}$$

$$= |\sigma\sqrt{2\pi}|e^{-\pi^2 k^2 2\sigma^2} \tag{20}$$

so then

$$\implies \mathscr{F}\Big(e^{-\frac{x^2+y^2}{2\sigma^2}}\Big) = (|\sigma\sqrt{2\pi}|e^{-\pi^2 k_x^2 2\sigma^2}) \cdot (|\sigma\sqrt{2\pi}|e^{-\pi^2 k_y^2 2\sigma^2}) \tag{21}$$

$$= 2\pi\sigma^2 e^{-2\pi^2\sigma^2(k_x^2+k_y^2)}. \tag{22}$$

Placing this back into the main equation and isolating $\tilde{T}(k_x, k_y, k_z)$ yields

$$\implies \tilde{T}(k_x, k_y, k_z) = \frac{\sigma^2 A_o e^{-2\pi i k_z z_0} e^{-2\pi^2\sigma^2(k_x^2+k_y^2)}}{ik_x v_x + 2\kappa\pi(k_x^2 + k_y^2 + k_z^2)}. \tag{23}$$

3 Inverse Transform Over z Direction

We first take the inverse transform of $\tilde{T}(k_x, k_y, k_z)$ with respect to k_z as

$$\tilde{T}(k_x, k_y, z) = \int_{-\infty}^{\infty} \tilde{T}(k_x, k_y, k_z) e^{2\pi i k_z z} dk_z \tag{24}$$

$$= \frac{\sigma^2 A_o e^{-2\pi^2\sigma^2(k_x^2+k_y^2)}}{2\kappa\pi} \int_{-\infty}^{\infty} \frac{e^{2\pi i k_z(z-z_0)}}{\frac{ik_x v_x}{2\kappa\pi} + (k_x^2 + k_y^2 + k_z^2)} dk_z \tag{25}$$

and if we let $p^2 = \frac{ik_x v_x}{2\kappa\pi} + k_x^2 + k_y^2$, then the denominator of the integrand is

$$k_z^2 + p^2 = (k_z + ip)(k_z - ip) \tag{26}$$

so the integral over k_z looks like

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i k_z (z - z_0)}}{(k_z + ip)(k_z - ip)} dk_z. \tag{27}$$

We can integrate this over appropriate contours in the complex plane.

3.1 Positive Imaginary Pole

We take the contour

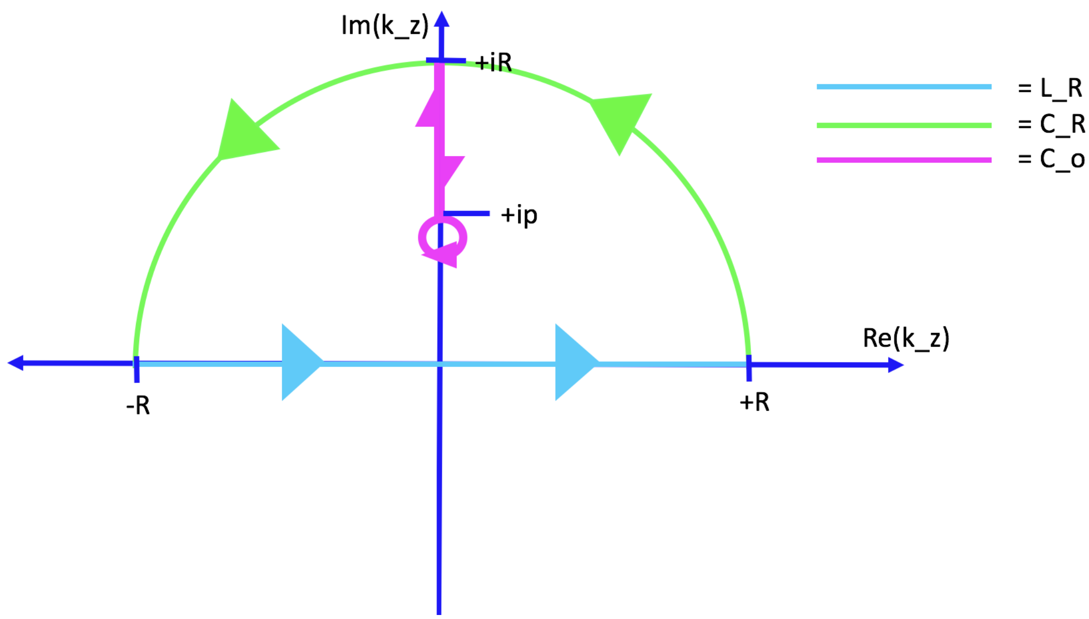

Figure 5

where

$$\mathbf{C} = C_R + C_o + L_R \tag{28}$$

$$\lim_{R\to\infty}\left(\int_{\mathbf{C}} f(k_z)dk_z = \int_{C_R} f(k_z)dk_z + \int_{C_o} f(k_z)dk_z + \int_{L_R} f(k_z)dk_z = 0\right) \tag{29}$$

because **C** is a closed loop in the complex plane that does not contain any poles. We integrate each term separately.

C_R:

$$\lim_{R\to\infty}\left(\int_{C_R} f(k_z)dk_z\right) = \lim_{R\to\infty}\left(\int_{C_R} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z\right) \tag{30}$$

On C_R,

$$k_z = Re^{i\phi} \quad | \quad 0 \le \phi \le \pi \tag{31}$$

$$\implies dk_z = Rie^{i\phi} \tag{32}$$

and implanting these substitutions into the integral above gives

$$\lim_{R\to\infty}\left(\int_0^\pi \Big(\frac{iR}{e^{2i\phi}R^2+p^2}\Big)e^{2\pi Re^{i\phi}(z-z_o)+\phi}d\phi\right). \tag{33}$$

As $R \to \infty$, the term in parentheses drops to zero and the exponential term is a sinusoidally oscillating forever, so the whole integral drops to 0. So

$$\lim_{R\to\infty}\left(\int_{C_R} f(k_z)dk_z\right) = 0. \tag{34}$$

C_o:

$$\lim_{R\to\infty}\left(\int_{C_o} f(k_z)dk_z\right) = \lim_{R\to\infty}\left(\int_{C_o}\frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z\right) \tag{35}$$

is the integral over a contour going in the clockwise direction.

It is always wise to convert all *closed* complex plane paths such that they are in the *counter-clockwise* direction before integrating (so that curling your right hand around the path makes your thumb point **out** of the page.)

The Residue Theorem may only be applied to such paths.

Just like swapping integration bounds in a 1-D integral means you have to put a minus sign in front of the whole integral as well, we can say that

$$\lim_{R\to\infty}\left(\int_{C_o}\frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z\right) = \lim_{R\to\infty}\left(-\int_{-C_o}\frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z\right) \tag{36}$$

$$= -2\pi i \lim_{k_z\to+ip}\left(\frac{e^{2\pi i(z-z_o)k_z}}{k_z+ip}\right) \tag{37}$$

$$= -\frac{\pi}{p}e^{-2\pi p(z-z_o)} \tag{38}$$

where the limit behavior of R never comes into play because the path C_o does not depend on R in the diagram!

L_R:

$$\lim_{R\to\infty}\left(\int_{L_R} f(k_z)dk_z\right) = \lim_{R\to\infty}\left(\int_{L_R} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z\right) \tag{39}$$

and on L_R, k_z is composed by only its real component - k_z is real valued - and goes from $-\infty$ to ∞ so the integral becomes

$$= \int_{-\infty}^{\infty} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z, \tag{40}$$

exactly the integral we want for a real-valued k_z in $(-\infty, \infty)$.

Now adding all three parts together, we get

$$0 - \frac{\pi}{p}e^{-2\pi p(z-z_o)} + \int_{-\infty}^{\infty} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z = 0 \tag{41}$$

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z = \frac{\pi}{p}e^{-2\pi p(z-z_o)} \tag{42}$$

is the evaluated integral over k_z using the semicircular contour in the upper complex plane.

3.2 Negative Imaginary Pole

We take the contour

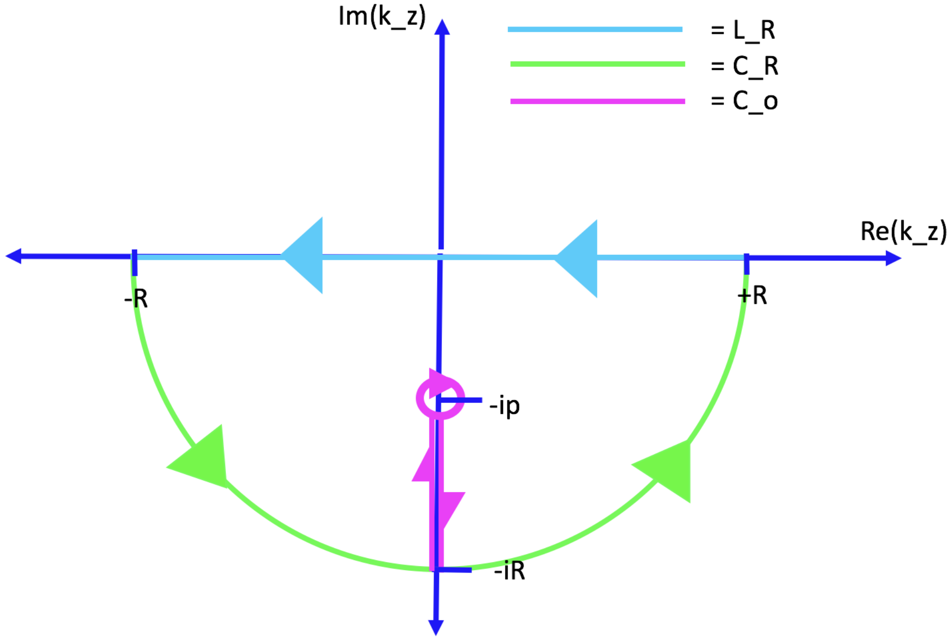

Figure 6

where again

$$\mathbf{C} = C_R + C_o + L_R \tag{43}$$

although the label labels apply to slightly different paths this time. So

$$\mathbf{C} = C_R + C_o + L_R \tag{44}$$

$$\lim_{R\to\infty}\left(\int_{\mathbf{C}} f(k_z)dk_z = \int_{C_R} f(k_z)dk_z + \int_{C_o} f(k_z)dk_z + \int_{L_R} f(k_z)dk_z = 0\right) \tag{45}$$

and now we treat each integral separately again.

C_R:

$$\lim_{R\to\infty}\left(\int_{C_R} f(k_z)dk_z\right) = \lim_{R\to\infty}\left(\int_{C_R} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z\right) \tag{46}$$

On C_R,

$$k_z = Re^{i\phi} \quad | \quad \pi \leq \phi \leq 2\pi \implies dk_z = Rie^{i\phi} \tag{47}$$

$$\implies \lim_{R\to\infty}\left(\int_{\pi}^{2\pi} \Big(\frac{iR}{e^{2i\phi}R^2+p^2}\Big)e^{2\pi Re^{i\phi}(z-z_o)+\phi}d\phi\right). \tag{48}$$

As $R \to \infty$, the term in parentheses drops to zero and the complex exponential term oscillates forever, so the whole integrand goes to 0. Thus

$$\lim_{R\to\infty}\left(\int_{C_R} f(k_z)dk_z\right) = 0. \tag{49}$$

C_o:

$$\lim_{R\to\infty}\left(\int_{C_o} f(k_z)dk_z\right) = \lim_{R\to\infty}\left(\int_{C_o} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z\right) \tag{50}$$

is the integral over a contour going in the clockwise direction, which we immediately rewrite as

$$= \lim_{R\to\infty}\left(-\int_{-C_o} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z\right) \tag{51}$$

$$= -2\pi i \lim_{k_z\to -ip}\Big(\frac{e^{2\pi i(z-z_o)k_z}}{k_z-ip}\Big) \tag{52}$$

$$= \frac{\pi}{p}e^{2\pi p(z-z_o)} \tag{53}$$

where the limit behavior of R never comes into play because the path C_o does not depend on R in the diagram.

L_R:

$$\lim_{R\to\infty}\left(\int_{L_R} f(k_z)dk_z\right) = \int_{\infty}^{-\infty} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z \tag{54}$$

because the our contour now has L_R going from $+\infty$ to $+\infty$. Then

$$= -\int_{-\infty}^{\infty} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z, \tag{55}$$

which is the negative of the integral we want for a real-valued k_z in $(-\infty, \infty)$.

Now adding all three parts together, we get

$$0 + \frac{\pi}{p}e^{2\pi p(z-z_o)} - \int_{-\infty}^{\infty} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z = 0 \tag{56}$$

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i k_z(z-z_0)}}{(k_z+ip)(k_z-ip)}dk_z = \frac{\pi}{p}e^{2\pi p(z-z_o)} \tag{57}$$

is the evaluated integral over k_z using the semicircular contour in the lower complex plane.

Notice how this contour yields an exponential that is growing, rather than decaying. If our value of $p \to \infty$, then this solution will blow up. Hence, this solution must only work for $p < 0$.

We now can combine the two contour solutions as

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i k_z (z-z_0)}}{(k_z+ip)(k_z-ip)} dk_z = \begin{cases} \frac{\pi}{p} e^{-2\pi p (z-z_o)} & p \geq 0 \\ \frac{\pi}{p} e^{2\pi p (z-z_o)} & p \leq 0 \end{cases}$$

$$= \frac{\pi}{p} e^{-2\pi |p| (z-z_o)} \quad \forall p \tag{58}$$

$$= \frac{\pi}{\sqrt{\frac{ik_x v_x}{2\kappa\pi} + k_x^2 + k_y^2}} e^{-2\pi \left| \sqrt{\frac{ik_x v_x}{2\kappa\pi} + k_x^2 + k_y^2} \right| (z-z_o)} \tag{59}$$

and inserting this into $\tilde{T}(k_x, k_y, z)$ gives

$$\tilde{T}(k_x, k_y, z) = \frac{\sigma^2 A_o e^{-2\pi^2\sigma^2(k_x^2+k_y^2)}}{2\kappa} \frac{e^{-2\pi \left| \sqrt{\frac{ik_x v_x}{2\kappa\pi} + k_x^2 + k_y^2} \right| (z-z_o)}}{\sqrt{\frac{ik_x v_x}{2\kappa\pi} + k_x^2 + k_y^2}}. \tag{60}$$

In order to satisfy the boundary condition, $T(z=0) = T_m$, we add an appropriate image and T_m, a perfectly adequate homogeneous solution to the differential equation we are trying to solve. So the temperature distribution in the lithospheric half space is described by

$$\tilde{T}(k_x, k_y, z) = \Big(\frac{\sigma^2 A_o e^{-2\pi^2\sigma^2(k_x^2+k_y^2)}}{2\kappa}\Big)\Big(\frac{e^{-2\pi \left| \sqrt{\frac{ik_x v_x}{2\kappa\pi} + k_x^2 + k_y^2} \right| (z-z_o)} + e^{2\pi \left| \sqrt{\frac{ik_x v_x}{2\kappa\pi} + k_x^2 + k_y^2} \right| (z+z_o)}}{\sqrt{\frac{ik_x v_x}{2\kappa\pi} + k_x^2 + k_y^2}}\Big) + T_m. \tag{61}$$

4 Inverse Transform Over x and y Directions

To obtain the final solution, we would have to compute the integral

$$T(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{T}(k_x, k_y, z) e^{2\pi i x k_x} e^{2\pi i y k_y} dk_x dk_y \tag{62}$$

and looking at $\tilde{T}(k_x, k_y, z)$ it can be deduced that differentiating with respect to z under the integral will yield something possibly more tractable. For simplicity, we will demonstrate the approach for the solution ignoring the image source. So

$$\frac{dT}{dz} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\tilde{T}}{dz} e^{2\pi i x k_x} e^{2\pi i y k_y} dk_x dk_y \tag{63}$$

$$= \frac{-\sigma^2 A_o \pi}{\kappa} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi^2\sigma^2(k_x^2+k_y^2)-2\pi(z-z_o)\sqrt{\frac{iv_x}{2\kappa\pi}k_x+k_x^2+k_y^2}+2\pi i x k_x+2\pi i y k_y} dk_x dk_y \tag{64}$$

and now it seems practical to transform this integral into polar coordinate representation before solving.

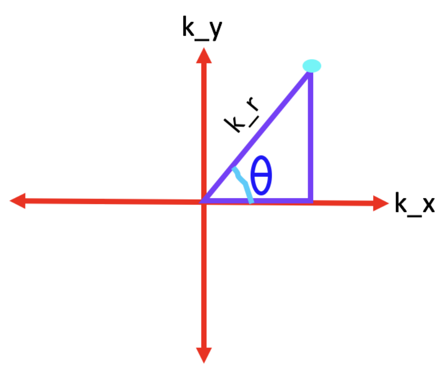

Figure 7

We treat $k_r^2 = k_x^2 + k_y^2$ and $(k_x, k_y) = (k_r\cos(\theta), k_r\sin(\theta))$ and with some rearranging, write

$$\frac{dT}{dz} = \frac{-\sigma^2 A_o \pi}{\kappa} \int_0^{\infty} e^{-\alpha k_r^2} k_r \Big(\int_0^{2\pi} e^{-\beta\sqrt{\lambda k_r\cos(\theta)+k_r^2}+\gamma k_r\cos(\theta)+\psi k_r\sin(\theta)} d\theta \Big) dk_r \tag{65}$$

where

- $\alpha = 2\pi^2\sigma^2$
- $\beta = 2\pi(z - z_o)$

- $\lambda = \frac{iv_x}{2\kappa\pi}$
- $\gamma = 2\pi ix$
- $\psi = 2\pi iy$.

The integral over θ can likely be solved using further complex analysis methods or by using Mathematica but we do not do that here.

We instead proceed to plot the temperature isotherms as time passes by and the heat flux as a function of depth,

$$q(k_x, k_y, z) = -k\frac{d\tilde{T}}{dz} = \frac{k_z\sigma^2 A_o\pi}{\kappa}e^{-2\pi^2\sigma^2(k_x^2+k_y^2)-2\pi(z-z_o)\sqrt{\frac{iv_x}{2\kappa\pi}k_x+k_x^2+k_y^2}}, \tag{66}$$

where k is the thermal conductivity in the Z direction, using

- $\sigma = 5000$
- $A_o = \frac{A}{\rho C} = 100$
- $\kappa = 10^{-7}$
- $z_o = 1000$
- $k_x = k_y = 2.5$
- $v_x =$ variable values.

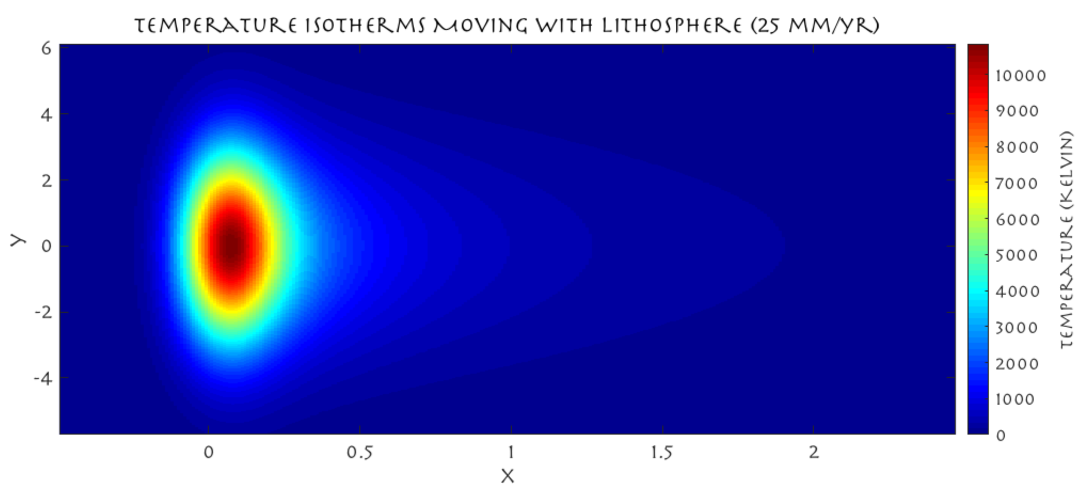

Figure 8

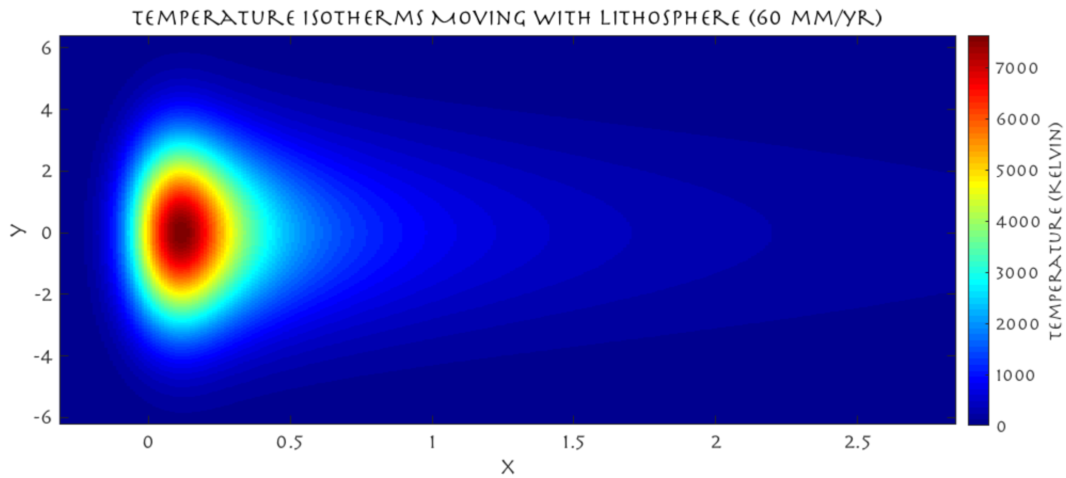

Figure 9

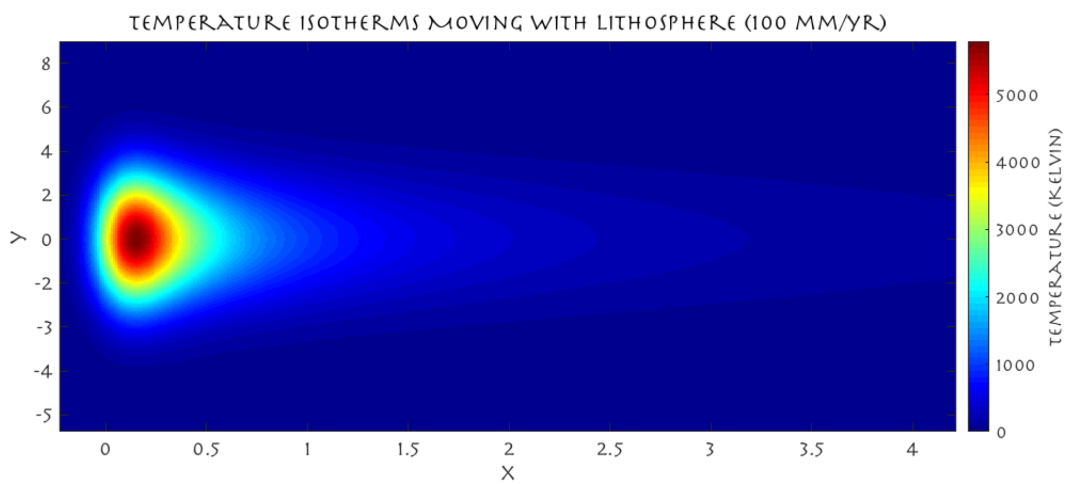

Figure 10