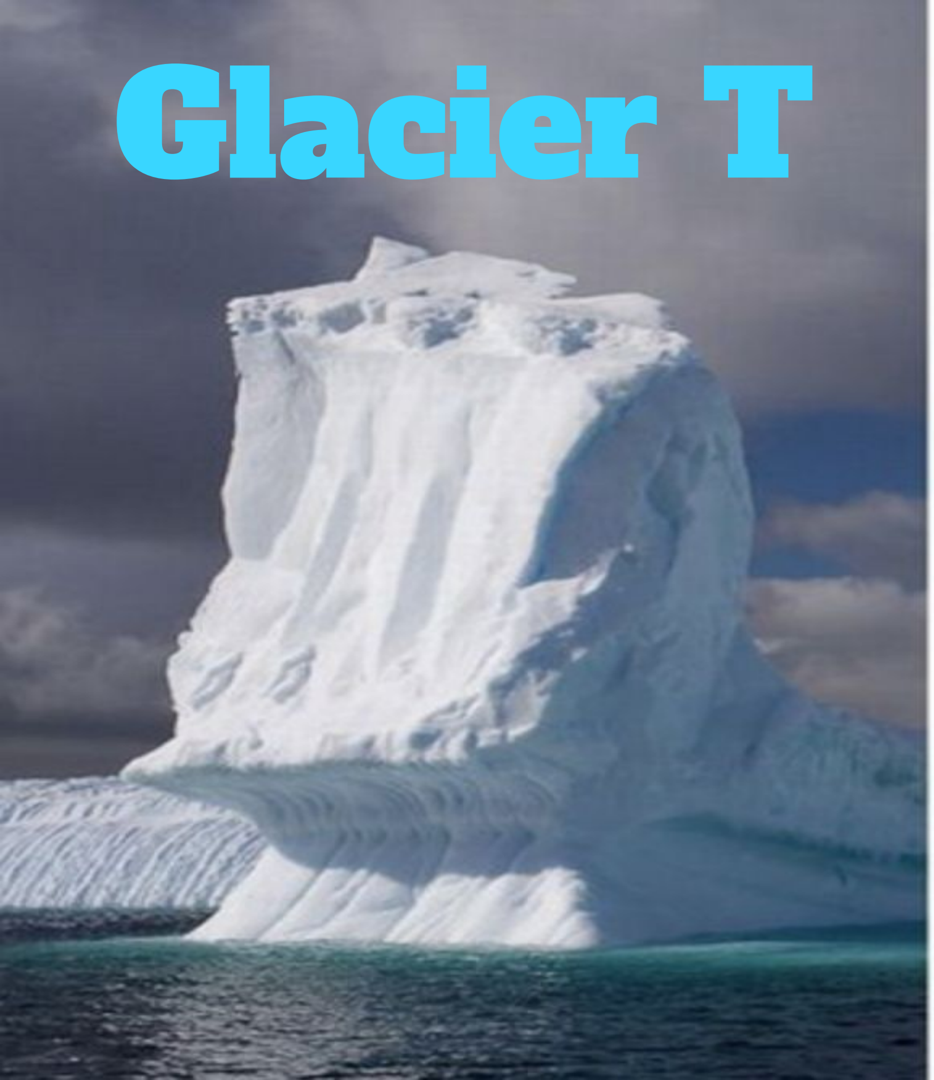
Glacier T



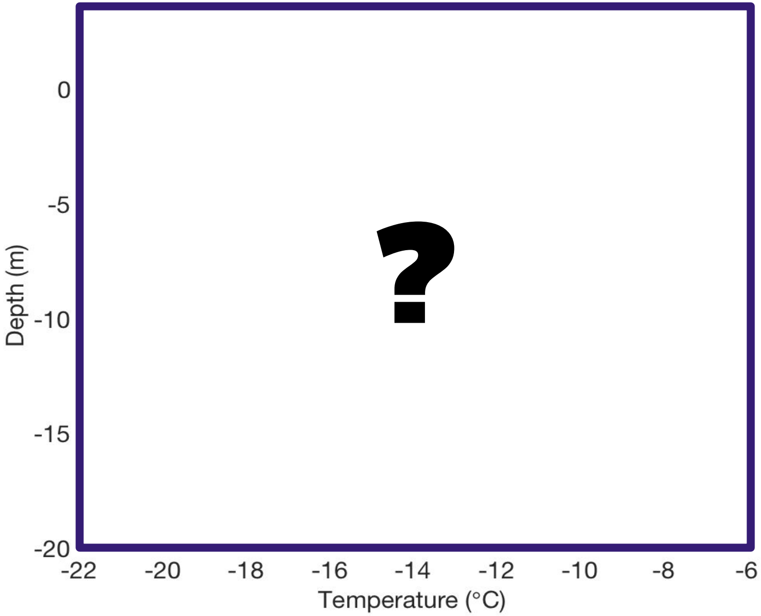
Nick Blanc & Margaret Morris

Problem

How does ice temperature close to a glacier surface vary with depth?

Things to consider:

- Seasonal temperature variations
- Heat diffusion dominates energy balance



Balance Equation

Energy Balance Equation $\longrightarrow$ Advection-Diffusion Equation:

$$\rho C \left(\frac{\partial T}{\partial t} + \underbrace{\mathbf{v} \nabla T}_{\text{advection}} \right) = \underbrace{\nabla (k \nabla T)}_{\text{diffusion}} + \underbrace{P}_{\text{production}} .$$

Simplified using constant k, and in 1-dimension:

$$\rho C \left(\frac{\partial T}{\partial t} + w \frac{dT}{dz} \right) = k \frac{d^2 T}{dz^2} + P.$$

Assumptions

Fourier law of heat diffusion

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial y^2}$$

We only care about one dimension (depth).

Assumptions

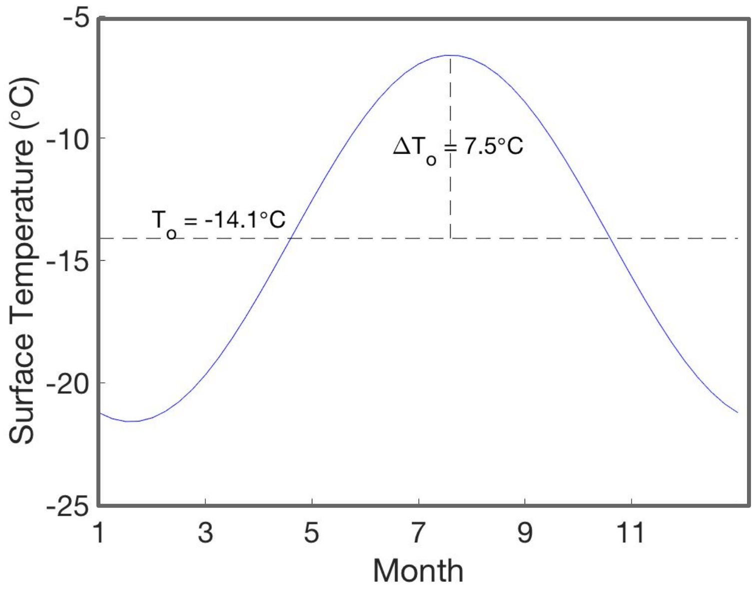
Boundary Conditions

- Temperature changes seasonally with some amplitude ΔT_0

$$T(0,t) = T_0 + \Delta T_0 \sin(\omega t)$$

- Constant at some depth (i.e. as y →∞)

$$T(\infty,t) = T_0$$



Derivation

Separate and conquer!

$$
\begin{aligned}
T(y,t) &= Y(y)R(t) \\
&= T_0 + Y_1(y)\cos(\omega t) + Y_2(y)\sin(\omega t).
\end{aligned}
$$

$$\kappa \frac{d^2 Y_1}{dy^2} = \omega Y_2$$

$$\kappa \frac{d^2 Y_2}{dy^2} = -\omega Y_1$$

$$\rightarrow \quad \frac{d^4 Y_2}{dy^4} + \left(\frac{\omega}{\kappa}\right)^2 Y_2 = 0 \quad \xrightarrow{\text{Guess!}} \quad Y_2 = C_0 e^{\alpha y}$$

Derivation

Assumed solution for Y_2

$$\boldsymbol{Y_2} = C_0 e^{\alpha y}$$

$$\alpha^4 + \left(\frac{\omega}{\kappa}\right)^2 = 0 \qquad \alpha = \pm\left(\frac{1 \pm i}{\sqrt{2}}\sqrt{\frac{\omega}{\kappa}}\right)$$

Derivation

$$Y_1(y) = \exp\left(-y\sqrt{\frac{\omega}{2\kappa}}\right)\left[b_3 \cos\left(y\sqrt{\frac{\omega}{2\kappa}} + b_4 \sin\left(y\sqrt{\frac{\omega}{2\kappa}}\right)\right]$$

$$Y_2(y) = \exp\left(-y\sqrt{\frac{\omega}{2\kappa}}\right)\left[b_1 \cos\left(y\sqrt{\frac{\omega}{2\kappa}} + b_2 \sin\left(y\sqrt{\frac{\omega}{2\kappa}}\right)\right]$$

Derivation

$$\kappa \frac{d^2 Y_1}{dy^2} = \omega Y_2$$

$$Ab_3 \sin\left(y\sqrt{\frac{\omega}{2\kappa}}\right) - Ab_4 \cos\left(y\sqrt{\frac{\omega}{2\kappa}}\right) = Ab_1 \cos\left(y\sqrt{\frac{\omega}{2\kappa}}\right) + Ab_2 \sin\left(y\sqrt{\frac{\omega}{2\kappa}}\right)$$

$$b_2 = b_3 = 0 \qquad b_1 = \Delta T_0 = -b_4$$

Derivation

$$T(y,t) = T_0 + Y_1\cos(\omega t) + Y_2\sin(\omega t)$$

$$= T_0 + \exp\left(-y\sqrt{\tfrac{\omega}{2\kappa}}\right)\cos(\omega t)\left[-\Delta T_0\sin\left(y\sqrt{\tfrac{\omega}{2\kappa}}\right)\right]$$

$$+ \exp\left(-y\sqrt{\tfrac{\omega}{2\kappa}}\right)\sin(\omega t)\left[\Delta T_0\cos\left(y\sqrt{\tfrac{\omega}{2\kappa}}\right)\right]$$

$$T(y,t) = T_0 + \Delta T_0\exp\left(-y\sqrt{\frac{\omega}{2\kappa}}\right)\sin\left(\omega t - y\sqrt{\frac{\omega}{2\kappa}}\right)$$

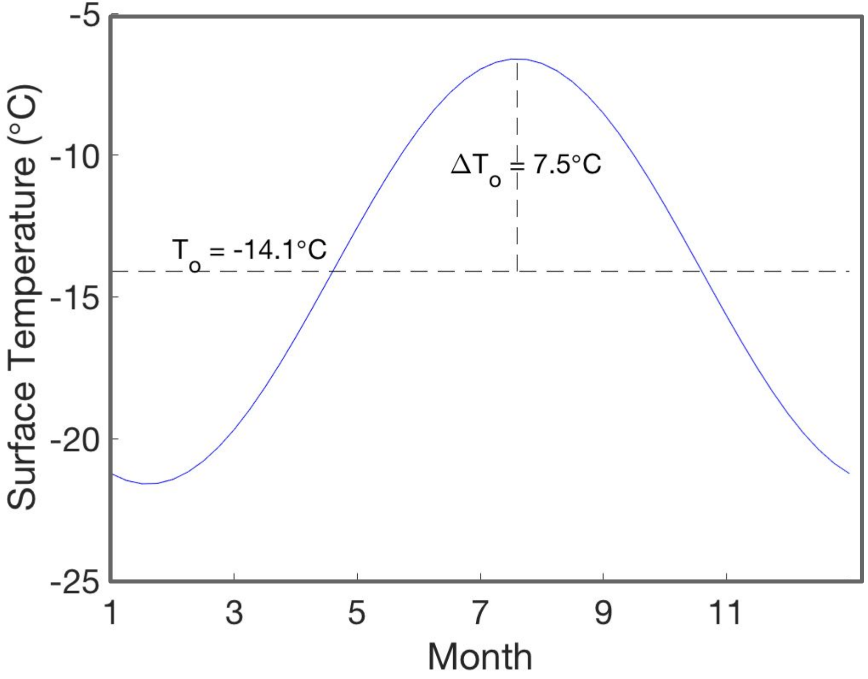
Glacier: Colle Gnifetti

Use borehole measurement temperatures from Haeberli & Funk 1991

Boundary Conditions:

$$T(0,t) = T_0 + \Delta T_0 \sin(\omega t)$$

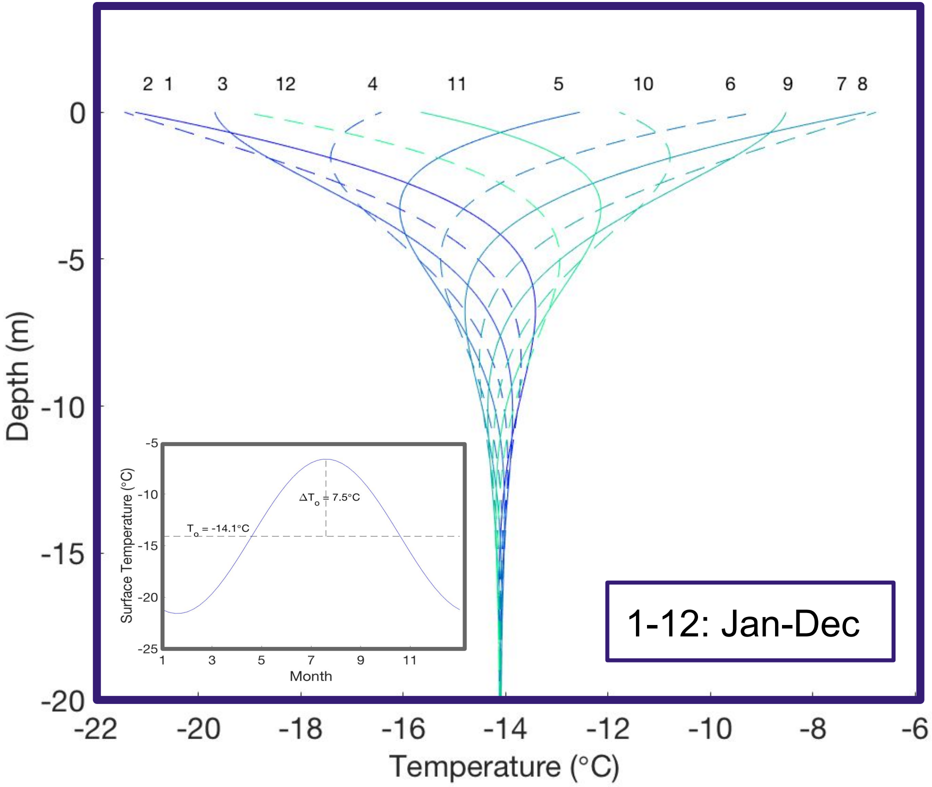
$$T(\infty,t) = T_0$$



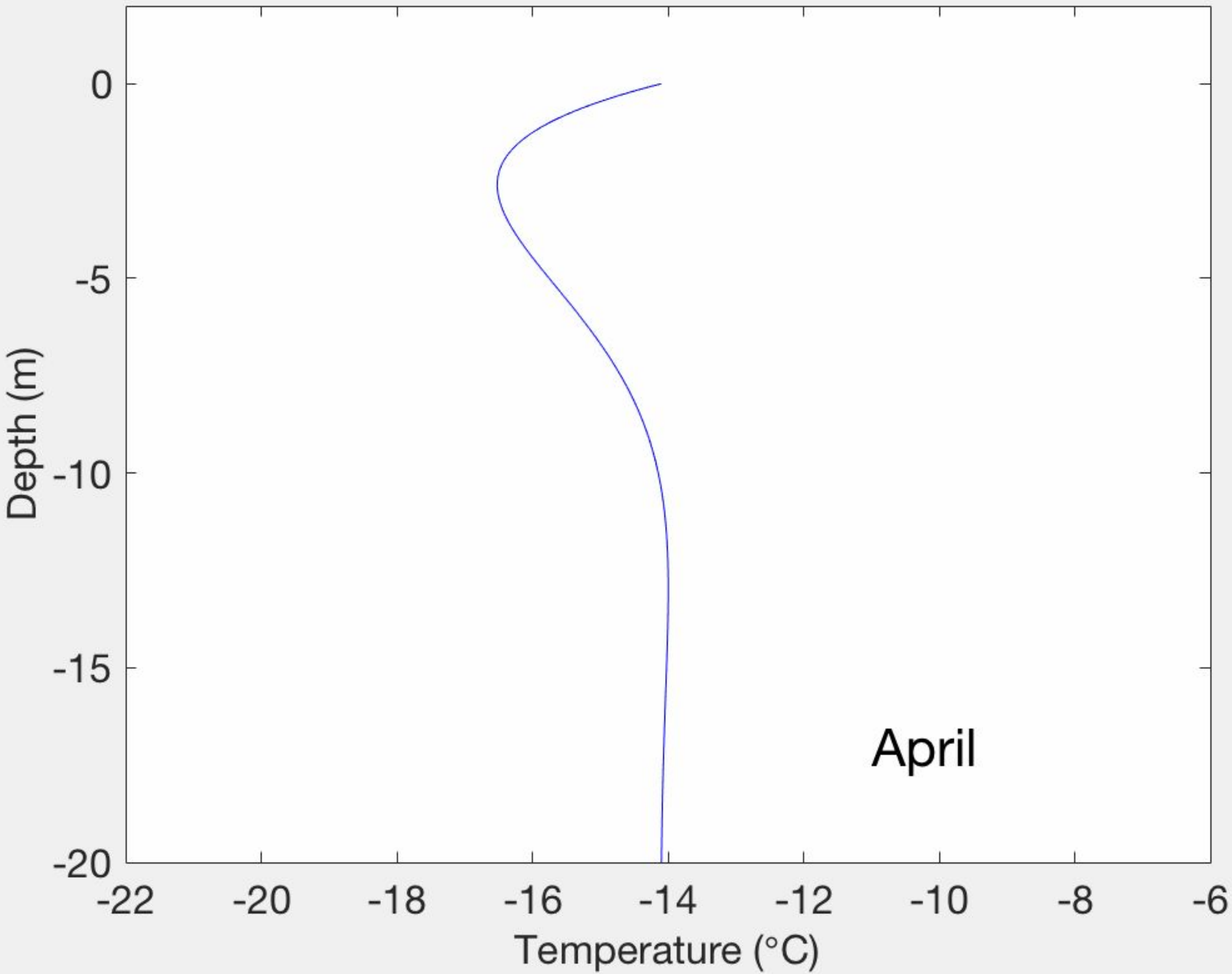
Result

Temperature varies with Depth

- Temp. variations small past 15m depth
- Temp settles to mean surface temperature at greater depths



$$T(y,t) = T_0 + \Delta T_0 \exp\left(-y\sqrt{\frac{\omega}{2\kappa}}\right)\sin\left(\omega t - y\sqrt{\frac{\omega}{2\kappa}}\right)$$


0
-5
-10
-15
-20
Depth (m)
-22
-20
-18
-16
-14
-12
-10
-8
-6
Temperature (°C)
April

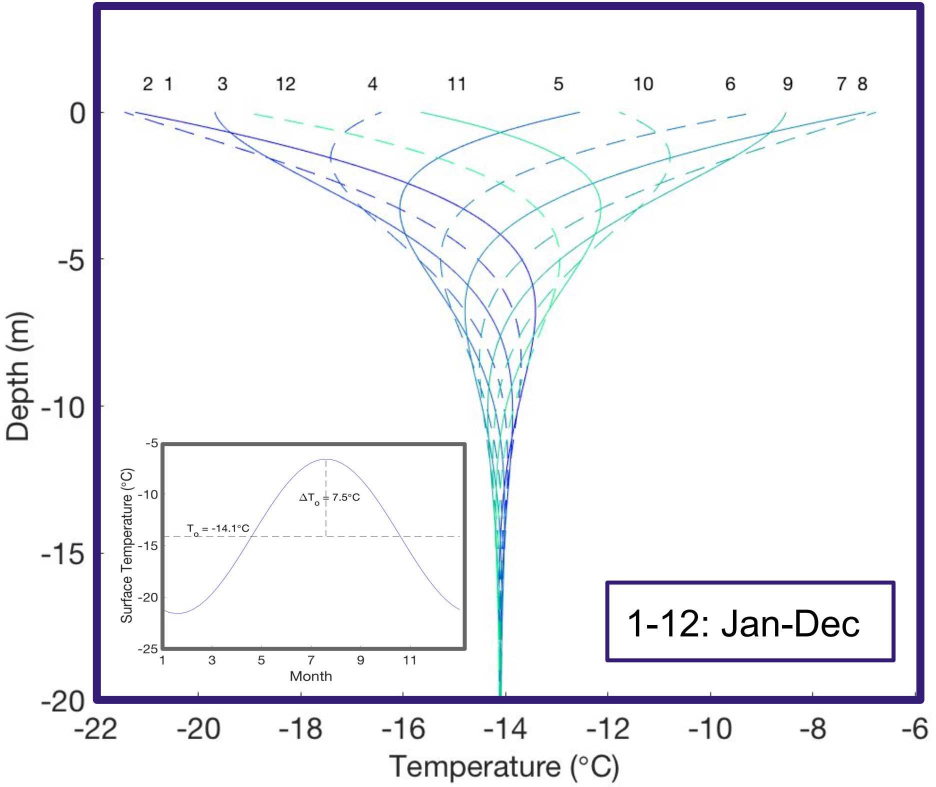
Deriving Temperature variation with Depth:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial y^2}$$

$$T(0,t) = T_0 + \Delta T_0 \sin(\omega t)$$

$$T(\infty,t) = T_0$$

$$T(y,t) = T_0 + \Delta T_0 \exp\left(-y\sqrt{\frac{\omega}{2\kappa}}\right) \sin\left(\omega t - y\sqrt{\frac{\omega}{2\kappa}}\right)$$



References

https://www.igsoc.org/journal/37/125/igs_journal_vol37_issue125_pg37-46.pdf

Journal of Glaciology, Vol. 37, No. 125, 1991

Borehole temperatures at the Colle Gnifetti core-drilling site (Monte Rosa, Swiss Alps)

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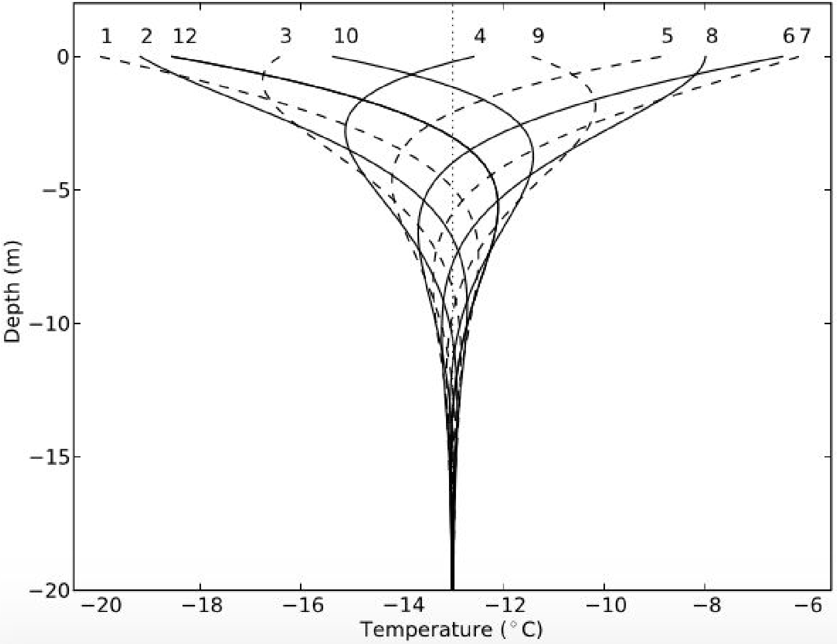
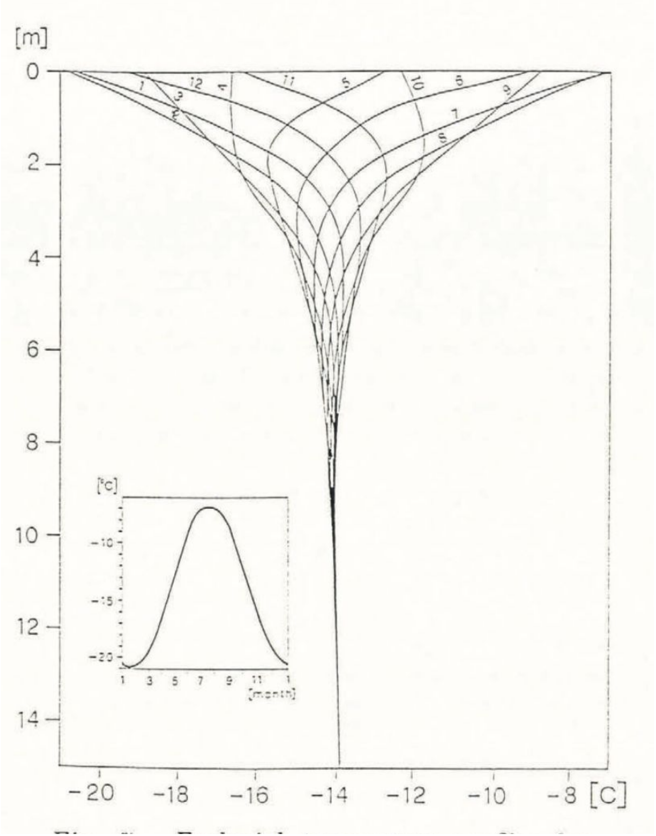
Cuffy Chapter 6

Extra Slides

Plot from Cuffy:

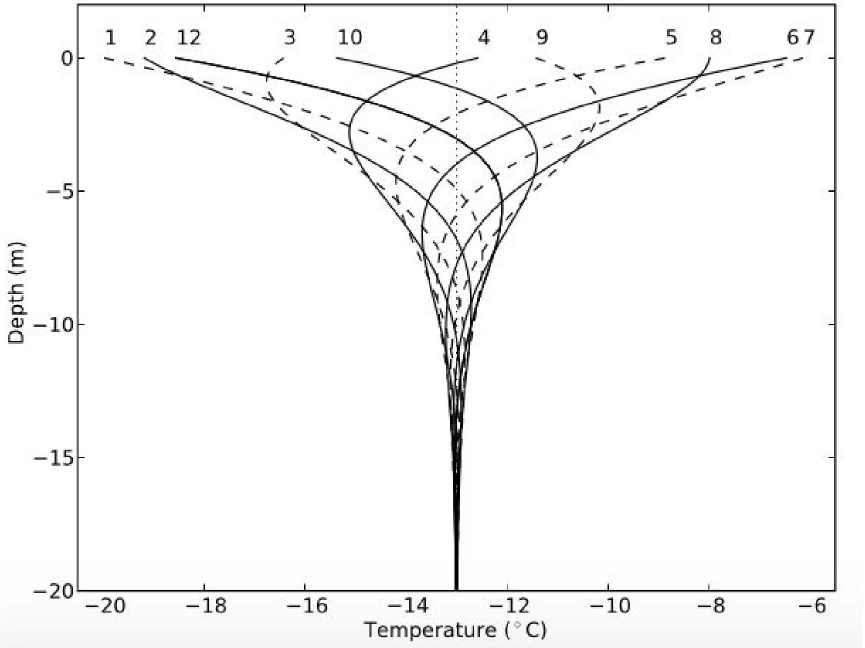


Plot from Haeberly & Funk:

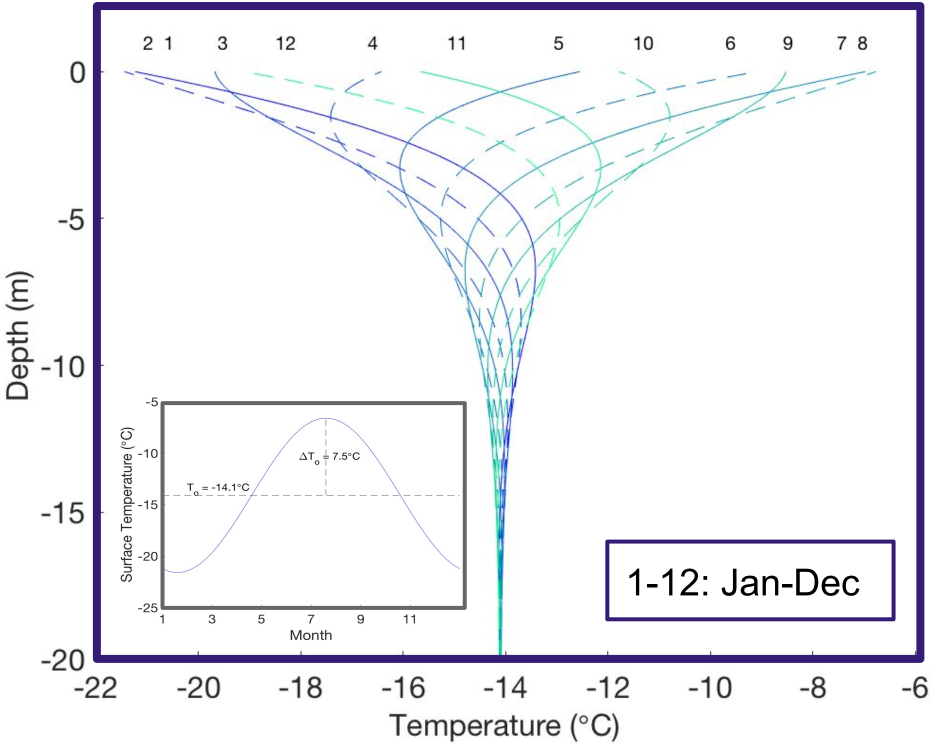


Extra Slides

Plot from Cuffy:



Plot we made:



Extra Slides

Plot from Haeberly & Funk:

Plot we made: