Postglacial Rebound

This lecture considers the classic problem of the viscous response of the mantle to rapid melting of the ice sheets following the last glacial maximum. The approach is similar to section 6-10 in *Turcotte and Schubert* [2002] but is for an arbitrary shaped initial topography rather than a single wavelength cosine function. The initial condition is shown in Figure 1.

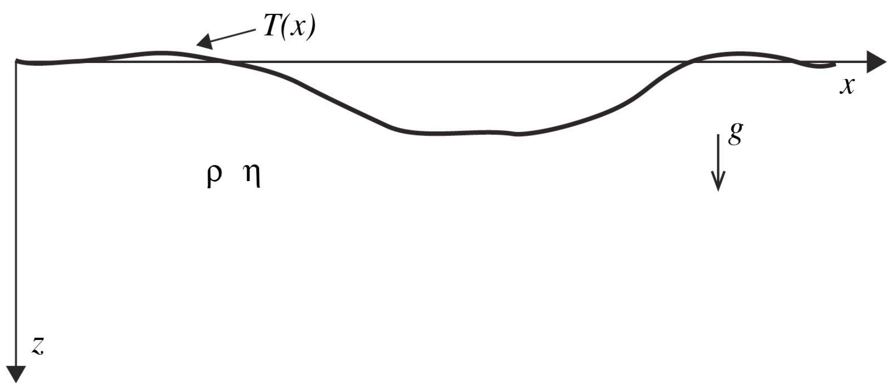

Figure 1. Viscous half space with an initial surface topography that decays exponentially with time under the restoring force of gravity.

The main parameters are:

$T(x) -$ initial topography (m)

$\eta -$ viscosity ($Pa\ s$)

$\rho -$ density ($kg\ m^{-3}$)

$g -$ acceleration of gravity ($m\ \mathrm{s}^{-2}$)

$u -$ x-velocity ($m\ s^{-1}$)

$w -$ z-velocity ($m\ s^{-1}$)

Guess at a Solution - Dimensional Analysis

We can make an initial guess at the time evolution of the topography $T(t)$ assuming a single wavelength λ for the initial surface topography $T(0)$. The guess is

$$T(t) = T(0)e^{-t/\tau_r} \ .$$

The relaxation time should increase as the viscosity increases and decrease as the restoring force increases so we put these in the numerator and demoninator, respectively $\frac{\eta}{\rho g}$. However the result has dimensions of *m s*. To make this into a time we can divide by the wavelength resulting in our initial guess at the relaxation time $\tau_r = \frac{\eta}{\rho g \lambda}$. We'll compare this with the relaxation time based on the full derivation.

Exact Solution

We'll assume the mantle is incompressible $\nabla \bullet \vec{u} = 0$ and the model is 2-D so this requires that

$$\frac{\partial u}{\partial x} = -\frac{\partial w}{\partial z} \ .$$

As discussed in T&S and during the last lecture, we can define a stream function $\psi(x,z)$ such that $u = -\frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial x}$.

This ensures that the material is incompressible $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{\partial^2 \psi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial z \partial x} = 0$.

The stresses can be related to the stream function as

$$\tau_{xx} = 2\eta \frac{\partial u}{\partial x} = -2\eta \frac{\partial^2 \psi}{\partial x \partial z}$$

$$\tau_{zz} = 2\eta \frac{\partial w}{\partial z} = 2\eta \frac{\partial^2 \psi}{\partial z \partial x}$$

$$\tau_{xz} = \eta \left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} \right)$$

The force balance equations become (i.e., equations 6.72 and 6.73 in T&S)

$$\eta \left(\frac{\partial^3 \psi}{\partial x^2 \partial z} + \frac{\partial^3 \psi}{\partial z^3} \right) + \frac{\partial P}{\partial x} = 0$$

$$\eta \left(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial z^2 \partial x} \right) - \frac{\partial P}{\partial z} = 0 .$$

Following T&S we take the derivative of the first equation with respect to z and the second equation with repect to x and add them to obtain the biharmonic equation.

$$\frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial z^2 \partial x^2} + \frac{\partial^4 \psi}{\partial z^4} = \nabla^4 \psi = 0$$

The boundary conditions for this problem are that the solution must vanish for large z, and the surface shear stress is zero.

$$\lim_{x \to \infty} \psi(x,z) = 0$$

$$\tau_{xz}\Big|_0 = 0$$

Also the surface pressure gradient is related to the topography gradient.

$$\left.\frac{\partial P}{\partial x}\right|_0 = -\rho g \frac{\partial T}{\partial x}$$

Note that uniform topography does not drive any flow. The flow is driven by the horizontal pressure gradient setup by the topography gradient. Take the Fourier transform of the biharmonic equation with respect to x to arrive at

$$(i2\pi k)^4 \psi + 2(i2\pi k)^2 \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^4 \psi}{\partial x^4} = 0 \ .$$

We guess a general solution of the form

$$\psi(k,z) = A(k)e^{-2\pi|k|z} + B(k)ze^{-2\pi|k|z} + C(k)e^{2\pi|k|z} + D(k)ze^{2\pi|k|z} \ .$$

The solution must go to zero for large z so $C = D = 0$ and the remaining terms are

$$\psi(k,z) = (A + Bz)e^{-2\pi|k|z} .$$

Next we use the boundary condition that the shear traction must vanish at the surface.

$$\tau_{xz}\Big|_0 = \eta\left.\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2}\right)\right|_0 = 0$$

We need to compute these derivatives

$$\frac{\partial^2 \psi}{\partial x^2} = -(2\pi k)^2 \psi \ , \ \frac{\partial \psi}{\partial z} = \left[B - 2\pi|k|(A + Bz)\right]e^{-2\pi|k|z} \text{ and}$$

$$\frac{\partial^2 \psi}{\partial z^2} = \left[-2B(2\pi|k|) + (2\pi|k|)^2 (A + Bz)\right]e^{-2\pi|k|z} .$$

The boundary condition becomes

$$\left.\frac{\tau_{xz}}{\eta}\right|_0 = -2(2\pi|k|)^2 A + 2(2\pi|k|)B = 0 \quad \text{so} \quad B = 2\pi|k|A$$

The stream function and the two velocity components are

$$\psi(k,z) = A(1 + 2\pi|k|z)e^{-2\pi|k|z}$$

$$u = A(2\pi|k|)^2 z e^{-2\pi|k|z}$$

$$w = A(i2\pi k)(1 + 2\pi|k|z)e^{-2\pi|k|z} .$$

Finally we need to match the surface pressure gradient boundary condition $\left.\frac{\partial P}{\partial x}\right|_0 = -\rho g \frac{\partial T}{\partial x}$. From the force balance equation we have $\frac{\partial P}{\partial x} = -\eta\left(\frac{\partial^3 \psi}{\partial x^2 \partial z} + \frac{\partial^3 \psi}{\partial z^3}\right)$. But we know that $\left.\frac{\partial \psi}{\partial z}\right|_0 = 0$ so only one term remains. In the transform domain this boundary condition becomes $\eta \frac{\partial^3 \psi}{\partial z^3} = \rho g (i2\pi k) T(k)$. Taking the third derivative of the stream function and solving for A we find $A = \frac{(i2\pi k)\rho g T(k)}{2\eta(2\pi k)^3}$. Putting this result in to the equation for the vertical velocity we find $w(k,0) = \frac{-\rho g}{2\eta(2\pi|k|)} T(k)$. We also know that $w = \frac{\partial T}{\partial t}$ so we end up with a differential equation for the time evolution of the topography $\frac{\partial T}{\partial t} = -\frac{\rho g}{4\pi |k| \eta} T$. The solution to this differential equation is $T(k,t) = T(k,0) e^{-\frac{\rho g}{4\pi |k| \eta} t}$. From this we find the characteristic relaxation time is $\tau_r = \frac{4\pi\eta}{\rho g \lambda}$. Note that this exact solution differs from the initial guess by 4π which is OK for a first order calculation.

Lets consider the example of Fennoscandia which has a characteristic wavelength of 3000 *km* a mantle density of 3300 *kg* m^{-3} and a characeristic relaxation time of 4400 *yr*. Using the formula we arrive at a viscosity of 1.1×10^{21} *Pa s*.

References

Turcotte, D. L., and G. Schubert, *Geodynamics: Second Edition*. Cambridge, UK: Cambridge University Press, 2002.