CHAPTER 3

1

Rearrange (3.12.2) to make κ the subject, substitute for κ into (3.12.1) and rearrange this into a quadratic in m^2:

$$m^4 - \epsilon' m^2 - \frac{\epsilon''^2}{4} = 0$$

This can be solved to obtain the required result for m. To find the corresponding expression for κ, we can follow a very similar procedure but this time obtaining a quadratic in κ^2.

2

Substitute the result for κ from problem 1 into equation (3.15). Introduce the variables

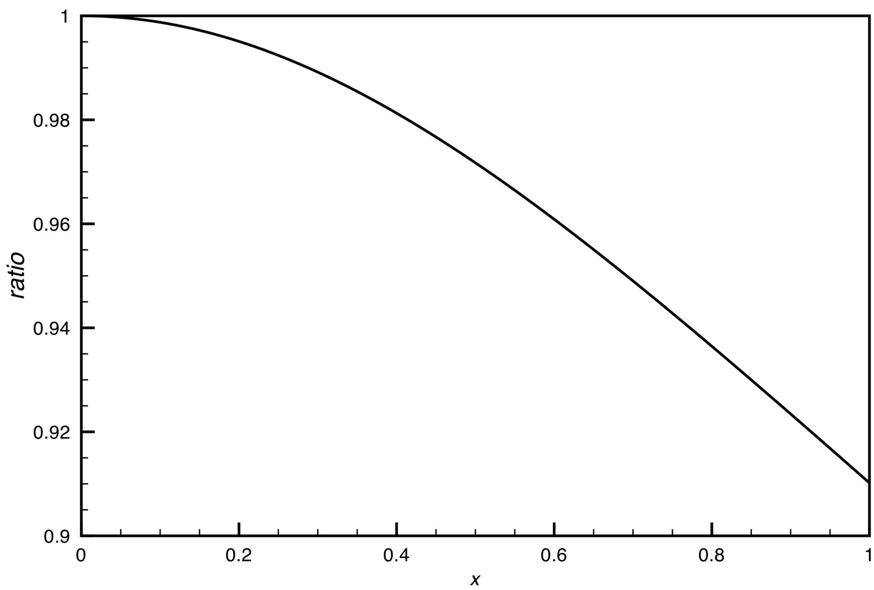
$\lambda_0 = 2\pi c/\omega$ and $x = \epsilon''/\epsilon'$, giving

$$l_a = \frac{\sqrt{2}\lambda_0}{4\pi\sqrt{\epsilon'}} \frac{1}{\sqrt{\sqrt{1+x^2}-1}} \quad .$$

Thus the ratio of the approximate expression for l_a to the accurate expression is

$$\frac{\sqrt{2\left(\sqrt{1+x^2}-1\right)}}{x}$$

This function is plotted below. It is simple to verify that it does not fall below 0.99 provided that x does not exceed about 0.28.



3

Using the data given in the problem, the dielectric constant of sea water is 88.2-719i at 100 MHz. This gives κ = 17.8 (see problem 3.1) and hence (from equation 3.15) l_a = 13 mm. Repeating the calculation for a frequency of 100 kHz gives κ = 600 and l_a = 0.4 m. Note that when $\epsilon' \ll \epsilon''$ and $\epsilon'' = \sigma / 2\pi\epsilon_0 f$ (as here), we have

$$l_a \approx \sqrt{\frac{c^2 \epsilon_0}{4\pi\sigma f}} \quad .$$

Significant absorption lengths in sea water are only possible at very low frequencies (which is why VLF has to be used for radio communication with submerged submarines).

4

We can write the dielectric constant of (3.24) as

$$\epsilon_r = 1 - \frac{a}{\omega^2}$$

and so the wave velocity (phase velocity) is given by

$$v = \frac{c}{\sqrt{\epsilon}} = c\left(1 - \frac{a}{\omega^2}\right)^{-1/2} \quad .$$

The wave velocity can also be written as

$$v = \frac{\omega}{k}$$

so we have

$$k = \frac{\omega}{c}\left(1 - \frac{a}{\omega^2}\right)^{1/2} = \frac{1}{c}\left(\omega^2 - a\right)^{1/2} \quad .$$

Thus the group velocity is given by

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{\omega}{c}\left(\omega^2 - a\right)^{-1/2}$$

and hence

$$v_g = c\left(1 - \frac{a}{\omega^2}\right)^{1/2} \quad .$$

Multiplying the expressions for v and v_g gives the required result.

5

The following Octave code calculates the values of rpar and rperp, the perpendicular and parallel amplitude reflection coefficients, using equations (3.32.1) and (3.32.3):

```
>>> er=63.1-32.1*i
er =  63.100 - 32.100i
>>> tr=83*pi/180
tr =  1.4486
>>> d1=sqrt(er-(sin(tr))^2)
d1 =  8.1251 - 1.9754i
>>> rperp=(cos(tr)-d1)/(cos(tr)+d1)
rperp = -0.9720486 + 0.0066951i
>>> rpar=(d1-er*cos(tr))/(d1+er*cos(tr))
rpar = -0.015873 + 0.116547i
```

They are then converted to amplitude and phase form:

```
>>> rperpamp=abs(rperp)
rperpamp =  0.97207
>>> rperppha=arg(rpar)
rperppha =  3.1347
>>> rparamp=abs(rperp)
rparamp =  0.11762
>>> rparpha=arg(rperp)
rparpha =  1.7062
```

Thus the reflection coefficients can be written as $0.972\,e^{3.135i}$ and $0.118\,e^{1.706i}$ respectively. If we define the *x*-axis of the reflected radiation to correspond to the perpendicularly polarised component, the Stokes vector of the reflected radiation becomes, following (2.13), [0.958, 0.931, 0.032, -0.226] relative to the incident radiation. This is almost 100% polarised (the incidence angle is close to the Brewster angle) and mainly linearly polarised but with a small circularly polarised component.

6

In section 3.3.2 it is shown that the diffuse albedo is given by

$$r_d = 2\int_0^{\pi/2} |r(\theta_0)|^2 \cos\theta_0 \sin\theta_0 \, d\theta_0$$

where $r(\theta_0)$ is the amplitude reflection coefficient for radiation with an incidence angle of θ_0. It is convenient to make the substitution $\mu = \cos\theta_0$, so the expression becomes

$$r_d = 2\int_0^1 |r(\mu)|^2 \mu \, d\mu \quad .$$

The appropriate value of $r(\mu)$ is given by (3.33.1) as

$$r(\mu) = \frac{\mu - \sqrt{n^2 + \mu^2 - 1}}{\mu + \sqrt{n^2 + \mu^2 - 1}}$$

so we have

$$r_d = 2\int_0^1 \left(\frac{\mu - \sqrt{n^2 + \mu^2 - 1}}{\mu + \sqrt{n^2 + \mu^2 - 1}} \right)^2 \mu \, d\mu \quad .$$

The indefinite integral is

$$\frac{\mu^2}{3}\left(\frac{8\mu^4}{(n^2-1)^2} + \frac{12\mu^2}{n^2-1} + \frac{8\mu(n^2+\mu^2-1)^{3/2}}{(n^2-1)^2} + 3 \right)$$

so the definite integral is

$$\frac{1}{3}\left(\frac{8}{(n^2-1)^2} + \frac{12}{n^2-1} + \frac{8n^3}{(n^2-1)^2} + 3 \right) \quad .$$

This can be simplified to

$$\frac{3n^2 - 2n - 1}{3(n-1)^2}$$

as required. As n tends to 1, this expression has a limiting value of zero which is what we expect since there is no dielectric contrast. As n tends to infinity the expression tends to one, which is also what we expect since this case corresponds to perfect reflection.

7

Write

$$R = A\cos\theta_0\cos\theta_1$$

for the BRDF, where A is a constant to be determined. Substituting this into (3.41) to find the reflectivity at normal incidence gives

$$r(0,0) = 2\pi A\int_0^{\pi/2}\cos^2\theta_1\sin\theta_1\,d\theta_1 = \frac{2\pi A}{3}$$

and thus $A = 3/2\pi$. Hence, from (3.43), the diffuse albedo is given by

$$r_d = 6\int_0^{\pi/2}\cos^2\theta_0\sin\theta_0\,d\theta_0\int_0^{\pi/2}\cos^2\theta_1\sin\theta_1\,d\theta_1 = \frac{2}{3}$$

as required.

8

The reflectivity is given by

$$r = 2\pi\int_0^{\pi/2}R\cos\theta_1\sin\theta_1\,d\theta_1 = 2\pi A(\cos\theta_0)^{a-1}\int_0^{\pi/2}(\cos\theta_1)^a\sin\theta_1\,d\theta_1 \quad .$$

This can easily be evaluated by setting $\mu = \cos\theta_1$, so that

$$r = 2\pi A(\cos\theta_0)^{a-1}\int_0^{\pi/2}\mu^a\,d\mu = \frac{2\pi A}{a+1}(\cos\theta_0)^{a-1} \quad .$$

The diffuse albedo is given by

$$r_d = 2\int_0^{\pi/2}\cos\theta_0\sin\theta_0\,d\theta_0 = \frac{4\pi A}{a+1}\int_0^{\pi/2}(\cos\theta_0)^a\sin\theta_0\,d\theta_0 = \frac{4\pi A}{(a+1)^2} \quad .$$

Thus

$$r = \frac{r_d(a+1)}{2}(\cos\theta_0)^{a-1}$$

as required.

9

First we consider the stationary phase model. Condition (3.58.1) requires that

$$\cos\theta > \frac{1.58}{\pi} = 0.503$$

and hence that

$$\theta < 59.8^{\circ} \quad . \tag{a}$$

Condition (3.58.2) requires that

$$\frac{L}{\lambda} > \frac{6}{2\pi} = 0.95 \tag{b}$$

and condition (3.58.3) requires that

$$\frac{L}{\lambda} > \sqrt{\frac{17.3}{4\pi}} = 1.17 \tag{c}$$

Condition (3.58.2) is thus ineffective. Combining (c) with (3.57) gives

$$m < \frac{\sqrt{2}}{2\times 1.17} = 0.60 \tag{d}$$

(a) and (d) give the conditions specified in the question.

Now we consider the scalar model. Condition (3.60.1) requires that

$$\frac{L}{\lambda} = \frac{1}{2\times 0.18} = 2.78 \tag{e}$$

Condition (3.60.2) requires that

$$\frac{L}{\lambda} > \frac{6}{2\pi} = 0.95 \tag{f}$$

as for the stationary phase model, and condition (3.60.3) requires that

$$\frac{L}{\lambda} > \sqrt{\frac{17.3}{4\pi}} = 1.17 \quad , \tag{g}$$

again as for the stationary phase model. Of these three conditions, only (e) is effective. Combining this with (3.57) gives

$$m < \frac{\sqrt{2}}{2\times 2.78} = 0.25 \tag{h}$$

as required.

10

Combining equations (3.83) and (3.84) gives

$$\frac{\Delta f_{Doppler}}{\Delta f_{pressure}} = \frac{f R T}{\sigma N_A p c} \quad .$$

Substitute

$$R = k N_A$$

and

$$c = f \lambda$$

to obtain the required result.

11

The absorption coefficient is given by equation (3.87) as

$$\gamma_a = n \sigma_a$$

where n is the number density of absorbing particles and σ_a is the absorption cross-section. From the data given in the question, the absorption coefficient is 4×10^{-5} m^{-1}. The optical thickness due to absorption of a 1-km thick layer of this cloud would thus be only 0.04. However, to decide whether the cloud would be opaque or transparent we also need to consider the scattering coefficient. Since the droplets are very small (48 µm) compared with the wavelength of the radiation (15 mm), the scattering mechanism will be Rayleigh scattering. The dimensionless size parameter (equation 3.65) is

$$\chi = 0.020 \quad .$$

We don't really need to make a detailed calculation about this. The cross-sectional area of a droplet is 7.2×10^{-9} m^2, so the scattering cross-section of a droplet is very much smaller than this, and likely to be significantly smaller than the absorption cross-section. Thus we do not expect the cloud to be opaque.

We can make a detailed calculation to be sure. From equation (3.20) the dielectric constant of water at 20 GHz is 37.3-37.3 so the refractive index is 6.71-2.78i. Substituting these values into (3.70) gives the scattering efficiency as 5.8×10^{-8} so the scattering cross-section of a droplet is around 4×10^{-16} m^2. This is indeed much smaller than the absorption cross-section, so we were justified in concluding that scattering is negligible.

12

The first result follows very straightforwardly from substituting (3.92) into (3.91). If $hf << kT$, we can put

$$\frac{2hf^3}{c^2\left(e^{hf/kT}-1\right)} \approx \frac{2kTf^2}{c^2}$$

so the radiative transfer equation becomes

$$\frac{dL_f}{dz} = \gamma_a\left(\frac{2kTf^2}{c^2} - L_f\right) \quad .$$

Multiplying throughout by $c^2/2kf^2$ gives

$$\frac{d}{dz}\frac{c^2 L_f}{2kf^2} = \gamma_a\left(T - \frac{c^2 L_f}{2kf^2}\right)$$

so we see that

$$T_b = \frac{c^2 L_f}{2kf^2}$$

as required. This is the Rayleigh-Jeans approximation (e.g. equation 2.33)

13

The two-stream model is represented by equations (3.98.1) and (3.98.2) and we are looking for solutions such that D=1 when z=0, and both D and U are zero when z = -∞. We try solutions

$$D=\exp(\mu z)$$

$$U=U_0\exp(\mu z)$$

which satisfy the boundary conditions. Substituting these trial solutions into (3.98.1) gives

$$\mu U_0\exp(\mu z)=-(\gamma_a+\gamma_s)U_0\exp(\mu z)+\gamma_s\exp(\mu z)$$

and hence

$$U_0(\mu+\gamma_a+\gamma_s)=\gamma_s \qquad \text{(a)}$$

while substitution into (3.98.2) gives

$$\mu\exp(\mu z)=(\gamma_a+\gamma_s)\exp(\mu z)-\gamma_s U_0\exp(\mu z)$$

and hence

$$\mu-(\gamma_a+\gamma_s)=-\gamma_s U_0 \qquad \text{(b)}$$

Solving (a) and (b) simultaneously gives

$$\mu^2=(\gamma_a+\gamma_s)^2-\gamma_s^2=\gamma_a^2-2\gamma_a\gamma_s \qquad \text{(c)}$$

and

$$U_0=\frac{\gamma_a+\gamma_s-\mu}{\gamma_s} \qquad \text{(d)}$$

as required.

14

Following the argument presented for cloud on page 96 we estimate the scattering cross-section of a single ice crystal as 8 × 10^{-7} m^2. The number density of the ice crystals is 2 × 10^8 m^{-3}, so the scattering coefficient γ_s is of the order of 100 m^{-1}. Since the density of the snow is one tenth the density of ice, we estimate the absorption coefficient of snow as 10^{-3} m^{-1} and hence the ratio γ_2/γ_a is of the order of 10^5. Since this is very much greater than 1, we expect virtually all the light that enters the snowpack to be scattered out of it again. The attenuation length is roughly γ_s^{-1} ≈ 1 cm so provided the snowpack is more than a few centimetres deep it will be optically thick.