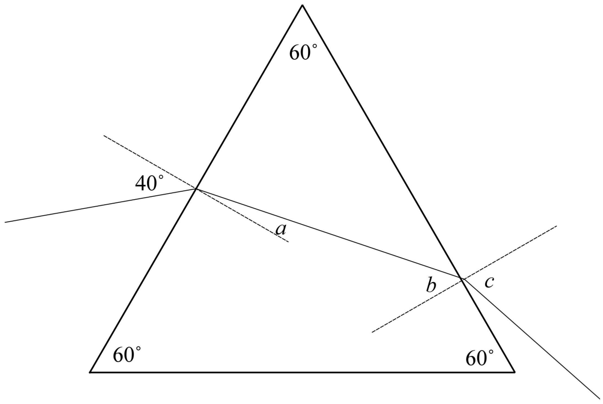
CHAPTER 6

1



We have

$$\sin a = \frac{\sin 40^\circ}{n}$$
$$b = 60^\circ - a$$
$$\sin c = n \sin b$$

and the total deviation of the ray is $c - 20^\circ$. For $n = 1.601$ we obtain

$$a = 23.671^\circ$$
$$b = 36.329^\circ$$
$$c = 71.524^\circ$$

giving a deviation of 51.524˚. For $n = 1.569$, the values are

$$a = 24.185^\circ$$
$$b = 35.815^\circ$$
$$c = 66.655^\circ$$

giving a deviation of 46.655˚. The range of deviations is thus 4.87˚.

2

From equation (2.24), the spectral irradiance at the surface can be written as

$$E_\lambda = L_{\lambda,S} \Delta\Omega \cos\theta$$

where $L_{\lambda,S}$ is the spectral radiance from the Sun, $\Delta\Omega$ is the solid angle subtended by the sun, and θ is the incidence angle of the radiation. Equation (3.45) shows that the BRDF of the surface is $1/\pi$ in all directions, so from equation (3.37) the reflected spectral radiance is

$$L_{\lambda,\mathrm{out}} = \frac{L_{\lambda,S} \Delta\Omega \cos\theta}{\pi} \quad .$$

Setting

$$L_{\lambda,S} = \frac{2hc^2 \epsilon_S}{\lambda^5 \left(e^{hc/\lambda k T_S} - 1 \right)}$$

where ϵ_S and T_S are, respectively, the sun's emissivity and temperature, and

$$L_{\lambda,\mathrm{out}} = \frac{2hc^2}{\lambda^5 \left(e^{hc/\lambda k T_b} - 1 \right)} \quad ,$$

where T_b is the brightness temperature of the reflected radiation, gives

$$e^{hc/\lambda k T_b} - 1 = \frac{\pi}{\epsilon_S \Delta\Omega \cos\theta} \left(e^{hc/\lambda k T_S} - 1 \right) \quad .$$

Substituting the values $\Delta\Omega = 6.76 \times 10^{-5}$ sr, $\epsilon_S = 0.99$ and $T_S = 5800$ K gives $T_b \approx 330$ K at 4 μm and 150 K at 10 μm.

3

Equation (6.6) shows that

$$T_1 = T_b e^{-\tau} + T_0\left(1 - e^{-\tau}\right)$$

so

$$e^{-\tau} = \frac{T_1 - T_0}{T_b - T_0}$$

Similarly equation (6.7) shows that

$$T_2 = T_b e^{-2\tau} + T_0\left(1 - e^{-2\tau}\right)$$

so

$$e^{-2\tau} = \frac{T_2 - T_0}{T_b - T_0} \quad .$$

Thus

$$\left(\frac{T_2 - T_0}{T_b - T_0}\right) = \left(\frac{T_1 - T_0}{T_b - T_0}\right)^2$$

which is easily rearranged to give the required result

$$T_b = \frac{\left(T_1 - T_0\right)^2}{T_2 - T_0} + T_0 \quad .$$

Substituting the values given in the problem yields $T_b = 285.25$ K.

(i) The sensitivities are found by differentiating:

$$\frac{\partial T_b}{\partial T_0} = \frac{\left(T_1 - T_0\right)\left(T_0 + T_1 - 2T_2\right)}{\left(T_2 - T_0\right)^2} + 1 \quad .$$

This is evaluated as 1.5625, so an uncertainty of ± 1 K in T_0 will give an uncertainty of about 1.6 K in the calculated value of T_b. Similarly,

$$\frac{\partial T_b}{\partial T_1} = \frac{2\left(T_1 - T_0\right)}{T_2 - T_0}$$

which we evaluate as 14. Thus the calculated value of T_b is almost ten times as to uncertainty in T_1 as to T_0.

(ii) The assumption that the Earth's curvature can be ignored does not introduce significant error into this calculation, as discussed on pages 117-8. However, the Rayleigh-Jeans approximation is not valid in the infrared region, and it is also not particularly reasonable to assume that the atmospheric temperature is uniform.

4

Differentiate equation (6.10) with respect to z and equate to (6.11), to give the thermal diffusion equation

$$\frac{\partial^2 T}{\partial z^2}=\frac{C\rho}{K}\frac{\partial T}{\partial t} \quad . \tag{a}$$

This can be solved conveniently by using complex exponential notation. Substituting a wave-like solution

$$T=A e^{i(\omega t-kz)} \tag{b}$$

into the diffusion equation gives

$$k^2=\frac{i\omega C\rho}{K}$$

and hence

$$k=\sqrt{\frac{\omega C\rho}{2K}}(1-i) \tag{c}$$

Substituting (b) into equation (6.10) gives

$$F=i k K A e^{i(\omega t-kz)}$$

so if we set

$$F_0=i k K A$$

we obtain the solutions

$$F=F_0 e^{i(\omega t-kz)}$$

and

$$T=\frac{F_0}{i k K}e^{i(\omega t-kz)} \quad .$$

Finally we substitute for k from (c) and take the real parts of the expressions to obtain the results we require:

$$F=F_0\cos\left(\omega t-z\sqrt{\frac{\omega C\rho}{2K}}\right)e^{-z\sqrt{\frac{\omega C\rho}{2K}}}$$

and

$$T=\frac{F_0}{\sqrt{\omega C\rho K}}\cos\left(\omega t-z\sqrt{\frac{\omega C\rho}{2K}}-\frac{\pi}{4}\right)e^{-z\sqrt{\frac{\omega C\rho}{2K}}} \quad .$$

5

The phase angle between the incoming solar flux and the surface temperature fluctuations is given by

$$\phi = \frac{2.3}{24} 2\pi = 0.602 \text{ radians}$$

and from equation (6.19) we see that this phase angle is given by

$$\cot\phi = 1 + \alpha\sqrt{\frac{2}{\rho c K \omega}} \quad .$$

Thus

$$\alpha\sqrt{\frac{2}{\rho c K \omega}} = 0.455 \qquad \text{(a)}$$

and hence substituting the other data given in the question, and taking $\omega = 7.29 \times 10^{-5}$ s^{-1}, we obtain

$$\rho c K = 4.01 \times 10^{6} \text{J}^{2} \text{m}^{-4} \text{s}^{-1} \text{K}^{-2}$$

(i.e. the thermal inertia is around 2000 in SI units).

Again using equation (6.19), we see that the magnitude of the ratio of the flux variations to the temperature variations is

$$\left(\alpha^2 + \alpha\sqrt{2\rho c K \omega} + \rho c K \omega\right)^{1/2} = \alpha\left(1 + \frac{\sqrt{2\rho c K \omega}}{\alpha} + \frac{\rho c K \omega}{\alpha^2}\right)^{1/2} \quad .$$

Using our result from (a), we find the ratio to be

$$\alpha\left(1 + \frac{2}{0.455} + \frac{2}{0.455^2}\right)^{1/2} = 3.88\,\alpha$$

so using the value of α given in the question, and taking the amplitude of the temperature fluctuations as 23 K, we find the amplitude of the flux fluctuations to be 491 K.

Finally, to interpret the value of α we can use equation (6.18), which shows that it corresponds to value of emissivity of close to 1 if the mean surface temperature is 290 K.

6

Substitute the expression for $\gamma(h)$ into equation (6.30):

$$\tau=\gamma_0\sqrt{\frac{R}{2}}\int_{h_0}^{\infty}\frac{e^{-\beta h}}{\sqrt{h-h_0}}\,dh \quad .$$

Now make the substitution

$$z=\sqrt{h-h_0}$$

to give

$$\tau=\gamma_0\sqrt{2R}\,e^{-\beta h_0}\int_0^{\infty}e^{-\beta z^2}\,dz \quad .$$

Recalling that

$$\int_0^{\infty}e^{-\beta z^2}\,dz=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}$$

gives

$$\tau=\gamma_0 e^{-\beta h_0}\sqrt{\frac{\pi R}{2\beta}}$$

from which the required result follows immediately.

Rearranging this expression to make h_0 the subject gives

$$h_0=\frac{1}{\beta}\ln\left(\frac{\gamma_0}{\tau}\sqrt{\frac{\pi R}{2\beta}}\right)$$

so substituting the values given in the question (and taking $\tau = 2.303$ and $R = 6378$ km) yields $h_0 =$ 5.0 km.

7

If the band 6 output is 145, we know that

$$16.03(L+3.2)+1=145$$

where L is the at-satellite radiance in W m^{-2} sr^{-1} µm^{-1}. Thus L = 5.78 W m^{-2} sr^{-1} µm^{-1}, and equation 6.9) gives the at-satellite brightness temperature as 269.8 K. The physical temperature of the surface is 273.15 K (since it consists of ice and water), and if we assume that the emissivity of the surface is very close to 1 this means that the at-surface brightness temperature of the surface is also 273.15 K. Thus the effect of atmospheric propagation is to contribute around -3.4 K to the measured brightness temperature. We expect typical atmospheric corrections to be a few kelvin, so this value is not surprising.