Linear Stability

7.1 Introduction

In this chapter we will be concerned with the onset of thermal convection when the fluid is heated just enough that weak convective motions begin to take over from conduction or radiation and transfer some of the heat. We will assume that the onset of convection occurs as a bifurcation from a motionless conductive state. We treat the onset of convection as a problem in the stability of the basic motionless state, i.e., we subject the basic state to perturbations of temperature and velocity and determine conditions under which the perturbations decay or amplify. The onset of convection corresponds to the state in which perturbations have zero growth rate.

If perturbations have a negative growth rate, i.e., if they decay, the basic motionless state is stable and heat is transported conductively. If perturbations have a positive growth rate, they amplify and establish a state of motion in which heat is partly transported convectively. We subject the basic state to perturbations of infinitesimal amplitude. In this case the stability problem can be linearized in the sense that quadratic and higher order products of perturbation quantities can be neglected compared to linear order perturbation quantities.

The linear stability or onset of convection problem is a classic problem with a large literature (see, e.g. Chandrasekhar, 1961). Although mantle convection is a highly nonlinear phenomenon, the linearized stability problem is relevant because it is amenable to analytical description and it contains much of the physics of the nonlinear convective state.

7.2 Summary of Basic Equations

The basic equations governing the onset of thermal convection are the conservation equations of mass, momentum, and energy discussed in Chapter 6. We adopt the Boussinesq approximation discussed in Chapter 6; the incompressible continuity equation is then the appropriate form of mass conservation

\[
\frac{\partial u_i^{*}}{\partial x_i^{*}} = 0
\]  

(7.2.1)

where \( u_i^{*} \) is the dimensionless fluid velocity in the perturbed state. We assume \( Pr = \infty \) as appropriate to the mantle and use (6.10.23) as the form of momentum conservation for a
Boussinesq fluid

\[ 0 = -\frac{\partial p^*}{\partial x_i^*} - g_i^* T^* Ra + \frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} \]  

(7.2.2)

In writing (7.2.2) we have simplified (6.10.23) by assuming constant dynamic viscosity \((\mu^* = 1)\) and we have used (7.2.1) to eliminate one of the viscous terms. All nondimensional quantities (denoted by asterisks) are scaled as in Chapter 6. The Rayleigh number \(Ra\) is defined in (6.10.20). Primes refer to perturbation quantities as defined in Chapter 6. With the assumptions enumerated above, the relevant energy equation is (6.10.33)

\[ \frac{DT^*}{Dt^*} = \frac{\partial T^*}{\partial t^*} + u_i^* \frac{\partial T^*}{\partial x_i^*} = \frac{\partial^2 T^*}{\partial x_i^* \partial x_i^*} + H^* \left( \frac{b^2 H_r \rho_r}{k_r \Delta T_r} \right) \]  

(7.2.3)

where density, specific heat at constant pressure, and thermal conductivity have all been taken as constant \((\bar{\rho}^* = 1, \bar{c}_p^* = 1, \bar{k}^* = 1)\).

The perturbation quantities in (7.2.1)–(7.2.3) are changes from a motionless adiabatic reference state (see Chapter 6). However, while the adiabatic state is an appropriate reference state for a vigorously convecting fluid, it is not a particularly good choice of a reference state for the onset of convection problem. The basic state at the onset of convective instability has a conductive temperature profile, not an adiabatic temperature profile. Since the adiabatic temperature of a Boussinesq fluid is a constant, \(T^*\) is essentially the total temperature of the fluid; \(T^*\) is not a perturbation quantity at all, in the sense that perturbation quantities have infinitesimal amplitude at the onset of convection. Instead \(T'\) is the sum of the motionless basic state steady conduction temperature profile \(T_c^*\) and a small amplitude departure \(\theta'^*\):

\[ \theta'^* \equiv T'^* - T_c^* \]  

(7.2.4)

The quantity \(\theta'^*\) is a true perturbation quantity and is dynamically induced.

Similarly, \(p'^*\) as defined in Chapter 6 is not a true perturbation quantity. The variable \(p'^*\) is the pressure perturbation relative to a hydrostatic pressure computed using the adiabatic density (Chapter 6); the adiabatic density of a Boussinesq fluid is a constant. In the basic state at the onset of convection there is a density variation associated with the conductive temperature profile; the density variation modifies the hydrostatic pressure at convective onset and \(p'^*\) includes this effect. In the motionless conductive state, \(u_i^* = 0, T'^* = T_c^*\) and (7.2.2) becomes

\[ \frac{\partial p'^*}{\partial x_i^*} = -g^* T_c^* Ra \]  

(7.2.5)

We denote the solution to this contribution to the hydrostatic pressure as \(p_c^*\) and define a true perturbation pressure \(\Pi'^*\) as

\[ \Pi'^* \equiv p'^* - p_c^* \]  

(7.2.6)

The quantity \(\Pi'^*\) is dynamically induced.

The equations for the onset of instability problem are obtained from (7.2.1)–(7.2.3) by substituting for \(T'^*\) from (7.2.4) and for \(p'^*\) from (7.2.6):

\[ \frac{\partial u_i^*}{\partial x_i^*} = 0 \]  

(7.2.7)
\[ 0 = -\frac{\partial \Pi^*}{\partial x_i^*} - g_i^* \theta^* Ra + \frac{\partial^2 u_i^*}{\partial x_i^*^2} \]  
(7.2.8)

\[ \frac{\partial \theta^*}{\partial t} + u_i^* \frac{dT_c^*}{dx_i^*} = \frac{\partial^2 \theta^*}{\partial x_i^*^2} \]  
(7.2.9)

In arriving at (7.2.9) we neglected the nonlinear advective heat transport term \( u_i^* \partial \theta^*/\partial x_i^* \) on the left side of (7.2.9) since this term is quadratic in the small-amplitude perturbations \( u_i^* \) and \( \theta^* \) (\( u_i^* \) as defined in Chapter 6 is a true perturbation quantity in the convective onset problems since the reference state in Chapter 6 is motionless as is the basic state at convection onset). In addition, to get (7.2.9) we used the equation satisfied by the basic state heat conduction temperature (see 6.12.7 and 6.10.8) with \( \tilde{k}^* = 1 \) and \( \tilde{\rho}^* = 1 \):

\[ 0 = \frac{\partial^2 T_c^*}{\partial x_i^*^2} + H^* \left( \frac{b^2 H_r \rho_r}{k_r \Delta T_r} \right) \]  
(7.2.10)

Finally, we simplify (7.2.9) by setting \( \partial \theta^*/\partial t^* = 0 \). This is partly justified by the fact that the onset of convection is defined by perturbations with zero growth rate. Additionally, the validity of \( \partial \theta^*/\partial t^* = 0 \) requires that the perturbations at the onset of convection are not oscillatory in time. The absence of time dependence at the onset of convection is known as the principle of exchange of stabilities (Chandrasekhar, 1961); in the remainder of this chapter we consider only those convective instability problems for which the principle is valid and we replace (7.2.9) by

\[ u_i^* \frac{dT_c^*}{dx_i^*} = \frac{\partial^2 \theta^*}{\partial x_i^*^2} \]  
(7.2.11)

The equations for the analysis of the linearized stability of conductive motionless states against convection are (7.2.7), (7.2.8), and (7.2.11). In the following sections we solve these equations for a number of different heating modes, boundary conditions, and geometries.

### 7.3 Plane Layer Heated from Below

We first consider the onset of thermal convection in an infinite horizontal fluid layer of thickness \( b \) heated from below (Lord Rayleigh, 1916). The problem is illustrated in Figure 7.1. The upper surface at \( y = 0 \) is maintained at temperature \( T_0 \) and the lower surface at \( y = b \) is maintained at temperature \( T_1 \); \( T_1 > T_0 \). The temperature scaling factor \( \Delta T_r \) (see Chapter 6) is taken as \( T_1 - T_0 \). The gravitational field acts in the +y-direction and the acceleration of gravity \( g \) is assumed constant. Thermally driven convection is expected because the fluid near the upper boundary is cooler and more dense than the fluid near the lower boundary; the fluid near the lower boundary will tend to rise and the fluid near the upper boundary will tend to sink.

In the absence of convection, heat will be conducted from the lower boundary to the upper boundary. With a constant thermal conductivity, the dimensionless conductive temperature \( T_c^* \) is a solution of (7.2.10) with \( H^* = 0 \). With the boundary conditions given above, \( T_c^* \) can only depend on \( y \), and integration of (7.2.10) gives the linear temperature profile

\[ T_c^* = \frac{T_0}{T_1 - T_0} + y^* \]  
(7.3.1)
where $y^* = y/b$. The corresponding heat flux is

$$ q_c^* = \frac{q_c}{k(T_1 - T_0)/b} = \frac{-k dT_c/dy}{k(T_1 - T_0)/b} = \frac{-dT_c^*}{dy^*} = -1 \quad (7.3.2) $$

or

$$ q_c = \frac{-k(T_1 - T_0)}{b} \quad (7.3.3) $$

The pressure in the motionless conductive basic state $\bar{p}$ is given by (6.10.2) as

$$ p = \bar{p} + p' \quad (7.3.4) $$

where $\bar{p}$ is the hydrostatic pressure associated with the adiabatic reference state density (see 6.10.3). This adiabatic density $\rho$ is a constant in the Boussinesq fluid layer and (6.10.3) becomes

$$ \frac{d\bar{p}}{dy} = \rho g \quad (7.3.5) $$

or

$$ \bar{p} = \rho g y + p_0 \quad (7.3.6) $$

where $p_0$ is the surface pressure. In the motionless state, $p' = p_c$, the hydrostatic pressure associated with the variation in the conductive temperature $T_c$. From (7.2.5) and (7.3.1) we can write

$$ \frac{dp_c^*}{dy^*} = -\left\{ \frac{T_0}{T_1 - T_0} + y^* \right\} Ra \quad (7.3.7) $$

which integrates to

$$ p_c^* = -Ra \left\{ \frac{T_0 y^*}{T_1 - T_0} + \frac{y^{*2}}{2} \right\} \quad (7.3.8) $$

(the constant of integration is zero). We can convert (7.3.8) to dimensional form using the scaling of Chapter 6 and the definition of the Rayleigh number in (6.10.20):

$$ p_c = -\rho g \alpha \left\{ T_0 y + \frac{1}{2} \left( \frac{T_1 - T_0}{b} \right) y^2 \right\} \quad (7.3.9) $$
The combination of (7.3.4), (7.3.6), and (7.3.9) gives the pressure in the motionless basic state as

\[ p = p_0 + \rho g y - \rho g \alpha \left\{ T_0 y + \frac{1}{2} \left( \frac{T_1 - T_0}{b} \right) y^2 \right\} \]  
(7.3.10)

The equations for the dynamically induced infinitesimal perturbations \( u^*, \pi^*, \theta^* \) are (7.2.7), (7.2.8), and (7.2.11). We limit the perturbation problem to two dimensions \((x^*, y^*)\) with \(x^*\) the horizontal coordinate, i.e., there are no spatial variations in the horizontal direction orthogonal to \(x^*\) and there are only two nonzero perturbation velocity components \((u^*, v^*)\) in the \((x^*, y^*)\) directions. We rewrite these equations as

\[ \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \]  
(7.3.11)

\[ \frac{\partial \Pi^*}{\partial x^*} = \left( \frac{\partial^2}{\partial x^*} + \frac{\partial^2}{\partial y^*} \right) u^* \]  
(7.3.12)

\[ \frac{\partial \Pi^*}{\partial y^*} = -\theta^* Ra + \left( \frac{\partial^2}{\partial x^*} + \frac{\partial^2}{\partial y^*} \right) v^* \]  
(7.3.13)

\[ v^* = \left( \frac{\partial^2}{\partial x^*} + \frac{\partial^2}{\partial y^*} \right) \theta^* \]  
(7.3.14)

The boundary conditions required for the solution of (7.3.11)–(7.3.14) comprise conditions on the temperature at the upper \((y^* = 0)\) and lower \((y^* = 1)\) surfaces and conditions on both velocity components \((u^*, v^*)\) at both surfaces. It is possible to consider a large variety of such conditions. For the moment we focus on boundaries that are isothermal, impermeable, and shear stress free; these conditions are expressed mathematically as

\[ v^* = \theta^* = 0 \quad \text{at} \quad y^* = 0, 1 \]  
(7.3.15)

\[ \frac{\partial u^*}{\partial y^*} = 0 \quad \text{at} \quad y^* = 0, 1 \]  
(7.3.16)

The number of equations can be reduced by introduction of the stream function \(\psi^*(6.3.2)\)

\[ u^* = \frac{\partial \psi^*}{\partial y^*}, \quad v^* = -\frac{\partial \psi^*}{\partial x^*} \]  
(7.3.17)

and by elimination of the pressure perturbation (differentiate (7.3.12) with respect to \(y^*\) and (7.3.13) with respect to \(x^*\) and subtract); (7.3.11)–(7.3.14) then become

\[ 0 = \nabla^4 \psi^* + Ra \frac{\partial \theta^*}{\partial x^*} = \frac{\partial^4 \psi^*}{\partial x^4} + 2 \frac{\partial^4 \psi^*}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi^*}{\partial y^4} + Ra \frac{\partial \theta^*}{\partial x^*} \]  
(7.3.18)

\[ 0 = \nabla^2 \theta^* + \frac{\partial \psi^*}{\partial x^*} = \left( \frac{\partial^2}{\partial x^*} + \frac{\partial^2}{\partial y^*} \right) \theta^* + \frac{\partial \psi^*}{\partial x^*} \]  
(7.3.19)
(see also 6.11.1 and 6.11.2). By differentiating (7.3.18) with respect to $x^*$ and substituting for $\frac{\partial \psi^*}{\partial x^*}$ from (7.3.19), we arrive at a single partial differential for $\theta^*$:

$$0 = \nabla^b \theta^* - Ra \frac{\partial^2 \theta^*}{\partial x^*}$$

$$= \frac{\partial^6 \theta^*}{\partial x^*^6} + 3 \frac{\partial^6 \theta^*}{\partial x^*^4 \partial y^*^2} + 3 \frac{\partial^6 \theta^*}{\partial x^*^2 \partial y^*^4} + \frac{\partial^5 \theta^*}{\partial y^*^5} - Ra \frac{\partial^2 \theta^*}{\partial x^*^2}$$  \hspace{1cm} (7.3.20)

The boundary conditions for (7.3.20) follow from a combination of (7.3.15)–(7.3.19) and are

$$\theta^* = \frac{\partial^2 \theta^*}{\partial y^*^2} = \frac{\partial^4 \theta^*}{\partial y^*^4} = 0 \text{ on } y^* = 0, 1$$  \hspace{1cm} (7.3.21)

Equation (7.3.20) is a linear partial differential equation with constant coefficients; the equation and the boundary conditions (7.3.21) are homogeneous. With $\Delta T_r = T_1 - T_0$, the Rayleigh number, defined in (6.10.20), can be written as

$$Ra = \frac{\alpha g (T_1 - T_0) b^3}{\kappa \nu}$$  \hspace{1cm} (7.3.22)

where $\kappa$ is the thermal diffusivity and $\nu$ is the kinematic viscosity.

The solution of (7.3.20) and (7.3.21) can be obtained by considering temperature perturbations that are periodic in the horizontal coordinate $x^*$:

$$\theta^* = \tilde{\theta}^* (y^*) \sin \left( \frac{2\pi x^*}{\lambda^*} \right)$$  \hspace{1cm} (7.3.23)

where $\lambda^* = \lambda / b$ is the a priori unknown dimensionless horizontal wavelength of the temperature perturbation. This approach is equivalent to a Fourier expansion of $\theta^*$ in the horizontal coordinate. Substitution of (7.3.23) into (7.3.20) yields a sixth-order ordinary differential equation for $\tilde{\theta}^* (y^*)$:

$$\left( \frac{d^2}{dy^*^2} - \frac{4\pi^2}{\lambda^*^2} \right)^3 \tilde{\theta}^* (y^*) = - \frac{4\pi^2}{\lambda^*^2} Ra \tilde{\theta}^* (y)$$  \hspace{1cm} (7.3.24)

The solution of this constant coefficient ordinary differential equation subject to the boundary conditions (from (7.3.21))

$$\tilde{\theta}^* = \frac{d^2 \tilde{\theta}^*}{dy^*^2} = \frac{d^4 \tilde{\theta}^*}{dy^*^4} = 0 \text{ on } y^* = 0, 1$$  \hspace{1cm} (7.3.25)

is

$$\tilde{\theta}^* = \tilde{\theta}_0^* \sin \pi y^*$$  \hspace{1cm} (7.3.26)

provided

$$\left( \pi^2 + \frac{4\pi^2}{\lambda^*^2} \right)^3 = \frac{4\pi^2 Ra}{\lambda^*^2}$$  \hspace{1cm} (7.3.27)
or

\[ Ra = Ra_{cr} = \frac{\left( \pi^2 + 4\pi^2/\lambda^*^2 \right)^3}{4\pi^2/\lambda^*^2} = \frac{\pi^4}{4\lambda^*^4} \left( 4 + \lambda^*^2 \right)^3 \]  \hspace{1cm} (7.3.28)

where \( Ra_{cr} \) is the critical Rayleigh number for the onset of convection.

The solution obtained for the temperature perturbation exists only for values of \( Ra \) given by (7.3.28). A plot of \( Ra_{cr} \) versus \( 2\pi/\lambda^* \) according to (7.3.28) is shown in Figure 7.2. It is seen that \( Ra_{cr} \) has a minimum value \( Ra_{cr}(\text{min}) \) given by solving

\[ \frac{\partial Ra_{cr}}{\partial (2\pi/\lambda^*)} = 0 \]  \hspace{1cm} (7.3.29)

The value of \( Ra_{cr}(\text{min}) \), the minimum critical Rayleigh number for the onset of convection, occurs at \( \lambda^* = \lambda_{cr}^* \); from (7.3.29) and (7.3.28), the values of \( \lambda_{cr}^* \) and \( Ra_{cr}(\text{min}) \) are

\[ \lambda_{cr}^* = 2\sqrt{2} = 2.828, \quad Ra_{cr}(\text{min}) = \frac{27\pi^4}{4} = 657.5 \]  \hspace{1cm} (7.3.30)

The value \( Ra_{cr}(\text{min}) \) is the smallest value of the Rayleigh number at which convection could occur in the infinite plane fluid layer. If \( Ra \) is smaller than \( Ra_{cr}(\text{min}) \) the conductive fluid layer is stable and no convection can occur. If \( Ra \) is greater than \( Ra_{cr}(\text{min}) \), the fluid layer is unstable and convection can occur within a range of wavelengths corresponding to \( Ra \geq Ra_{cr} \) (see 7.3.28 and Figure 7.2). The solution to this problem was first obtained by Lord Rayleigh (1916).

With the temperature perturbation given by a combination of (7.3.23) and (7.3.26)

\[ \theta^* = \hat{\theta}^* \sin \pi y^* \sin \frac{2\pi x^*}{\lambda^*} \]  \hspace{1cm} (7.3.31)

Figure 7.2. Critical Rayleigh number \( Ra_{cr} \) for the onset of thermal convection in a fluid layer heated from below with stress free boundaries as a function of dimensionless wave number \( 2\pi b/\lambda \).
one can determine $\psi^*$ from (7.3.19) as

$$
\psi^* = -\left(\frac{\lambda^*}{2\pi}\right) \left(\frac{4\pi^2}{\lambda^*} + \pi^2\right) \hat{\theta}_0^* \sin \pi y^* \cos \frac{2\pi x^*}{\lambda^*}
$$

(7.3.32)

From (7.3.17) and (7.3.32) the velocity components are

$$
u^* = -\left(\frac{4\pi^2}{\lambda^*} + \pi^2\right) \hat{\theta}_0^* \sin \pi y^* \sin \frac{2\pi x^*}{\lambda^*}
$$

(7.3.34)

The solution takes the form of counterrotating cells as illustrated in Figure 7.3. The width of each cell is $b\lambda^*/2$; for $\lambda_{cr}^* = 2\sqrt{2}$ this width is $b\sqrt{2}$. The aspect ratio of the cell at $\lambda = \lambda_{cr}^*$ is the ratio of cell width to height or $\sqrt{2}$. Because the system of equations and boundary conditions is linear and homogeneous, the stability analysis does not predict the amplitude of the convection, i.e., $\hat{\theta}_0^*$ is arbitrary.

Holmes (1931) applied this stability analysis to the Earth’s mantle. He concluded that the minimum critical thermal gradient for whole mantle convection is $3$ K km$^{-1}$ ($Ra_{cr}$ (min)) can be converted into a minimum critical thermal gradient for the onset of convection by setting $Ra = Ra_{cr}$ $(\text{min})$ in (7.3.22) and solving for $(T_1 - T_0)/b^*$. Since the near-surface thermal gradient is an order of magnitude larger than this value, he concluded that convection currents are required in the mantle and that these convection currents are responsible for continental drift. It took another 40 years before this revolutionary assertion became generally accepted.

For fixed-surface boundary conditions, i.e., $u^* = 0$ on $y^* = 0.1$ instead of the condition used above on $\partial u^*/\partial y^*$ (7.3.16), a numerical solution is required for the linear stability problem (Pellew and Southwell, 1940). For this case the minimum critical Rayleigh number is $Ra_{cr}$ $(\text{min}) = 1.707.8$ and the corresponding dimensionless wavelength of the convective rolls is $\lambda_{cr}^* = 2.016$. For a shear stress free upper boundary and a fixed lower boundary, $Ra_{cr}$ $(\text{min}) = 1.100.7$ and $\lambda_{cr}^* = 2.344$. Different velocity boundary conditions have about a factor of 2 effect on the value of the minimum critical Rayleigh number. As expected, fixed surface boundary conditions are stabilizing relative to free surface boundary conditions, i.e., the value of $Ra_{cr}$ $(\text{min})$ is higher with fixed boundaries. Table 7.1 summarizes these values of $Ra_{cr}$ $(\text{min})$ and $\lambda_{cr}^*$.

An alternative thermal boundary condition of relevance to mantle convection involves specification of the upward heat flux $q_b$ at the lower boundary instead of the temperature.
Table 7.1. Values of the Minimum Critical Rayleigh Number and Associated Dimensionless Horizontal Wavelength for the Onset of Convection in Plane Fluid Layers with Different Surface Boundary Conditions and Modes of Heating

<table>
<thead>
<tr>
<th>Surface Boundary Conditions and Mode of Heating</th>
<th>$Ra_{cr}$ (min)</th>
<th>$\lambda_{cr}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both boundaries shear stress free and isothermal, no internal heating. $H^* = 0$.</td>
<td>657.5</td>
<td>$2\sqrt{2} = 2.828$</td>
</tr>
<tr>
<td>Both boundaries fixed and isothermal. $H^* = 0$.</td>
<td>1,707.8</td>
<td>2.016</td>
</tr>
<tr>
<td>Shear stress free upper boundary, fixed lower boundary. both boundaries isothermal. $H^* = 0$.</td>
<td>1,100.7</td>
<td>2.344</td>
</tr>
<tr>
<td>Both boundaries shear stress free. upper boundary isothermal, lower boundary specified heat flux. $H^* = 0$.</td>
<td>384.7</td>
<td>3.57</td>
</tr>
<tr>
<td>Both boundaries fixed, upper boundary isothermal, lower boundary specified heat flux. $H^* = 0$.</td>
<td>1,295.8</td>
<td>2.46</td>
</tr>
<tr>
<td>Upper boundary shear stress free and isothermal, lower boundary fixed and heat flux prescribed. $H^* = 0$.</td>
<td>816.7</td>
<td>2.84</td>
</tr>
</tbody>
</table>

In this case the thermal boundary conditions are

$$\theta^{*'} = 0 \text{ on } y^* = 0 \text{ and } \frac{\partial \theta^{*'}}{\partial y^*} = 0 \text{ on } y^* = 1 \quad (7.3.35)$$

The velocity conditions on the boundaries can be taken as impermeable and either fixed (no-slip) or shear stress free. To obtain a temperature scale for these thermal boundary conditions we calculate the temperature drop across the fluid layer in the conduction state (the conduction temperature profile is linear in $y$, as in the case of isothermal boundaries):

$$\Delta T_r = \frac{q_b b}{k} \quad (7.3.36)$$

Substitution of (7.3.36) into (6.10.20) yields the form of the Rayleigh number for the heat flux lower boundary condition:

$$Ra_q = \frac{\alpha g q_b b^4}{k k v} \quad (7.3.37)$$

Solutions to this stability problem have been obtained by Sparrow et al. (1964). For shear stress free surface boundary conditions $Ra_{q,cr}(\text{min}) = 384.7$ and $\lambda_{cr}^* = 3.57$, for fixed surface boundary conditions $Ra_{q,cr}(\text{min}) = 1,295.8$ and $\lambda_{cr}^* = 2.46$, and for a shear stress free upper boundary and a fixed lower boundary $Ra_{q,cr}(\text{min}) = 816.7$ and $\lambda_{cr}^* = 2.84$. The values of the minimum critical Rayleigh numbers for prescribed lower boundary heat flux are somewhat smaller than those for the isothermal boundaries and the horizontal widths of the cells are larger. These results are summarized in Table 7.1.

If mantle convection is layered, then it would be appropriate to calculate the heat flow Rayleigh number for the upper mantle, since the upper mantle could be regarded as a separately convecting layer with a given heat flux at its base. To calculate $Ra_q$ for the upper mantle we require the mean heat flux $q_m$ through the upper mantle. The total heat flow from the interior of the Earth $Q$ can be obtained by the multiplication of the area of the continents by the mean continental heat flux and adding the product of the oceanic area and the mean oceanic heat flow. The continents, including the continental margins, have an
area \( A_c = 2 \times 10^8 \) km\(^2\). Multiplication of this by the mean observed continental heat flux of 65 mW m\(^{-2}\) gives the total heat flow from the continents as \( Q_c = 1.30 \times 10^{13} \) W. The oceans, including the marginal basins, have an area \( A_o = 3.1 \times 10^8 \) km\(^2\). Multiplication of this by the mean oceanic heat flux of 101 mW m\(^{-2}\) yields the total heat flow from the oceans as \( Q_o = 3.13 \times 10^{13} \) W. By adding the heat flow from the continents and the oceans, we find that the total surface heat flow is \( Q = 4.43 \times 10^{13} \) W. Division by the surface area \( A = 5.1 \times 10^8 \) km\(^2\) gives the mean surface heat flux of 87 mW m\(^{-2}\) (see Section 4.1.5).

A substantial fraction of the heat lost from the continents originates within the continental crust. This must be removed in order to obtain the mean heat flux through the upper mantle. We estimate that 37 mW m\(^{-2}\) of the continental heat flux is due to internal heat generation from radioactive elements in the crust. Thus the mantle contribution to the continental heat flux is 28 mW m\(^{-2}\). When this is multiplied by the area of the continents, including the margins, and added to the oceanic contribution, we find that the total heat flow from the mantle is \( Q_m = 3.69 \times 10^{13} \) W. Thus the mean heat flux through the upper mantle is \( q_m = 72.4 \) mW m\(^{-2}\) (see Section 4.1.5).

In order to complete the specification of the Rayleigh number \( Ra_q \) for upper mantle convection we take \( \rho = 3.6 \times 10^3 \) kg m\(^{-3}\), \( g = 10 \) m s\(^{-2}\), \( \alpha = 3 \times 10^{-5} \) K\(^{-1}\), \( b = 700 \) km, \( k = 4 \) W m\(^{-1}\) K\(^{-1}\), \( \kappa = 1 \) mm\(^2\) s\(^{-1}\), and \( \mu = 10^{21} \) Pas. Substitution of these values into (7.337) gives \( Ra_q = 4.8 \times 10^6 \). Thus the ratio of the Rayleigh number \( Ra_q \) to the minimum critical Rayleigh number for upper mantle convection \( Ra_{q,cr}(min) \) is between \( r = Ra_q/Ra_{q,cr}(min) = 3.6 \times 10^3 \) and \( 1.2 \times 10^4 \), depending upon the applicable velocity boundary conditions.

### 7.4 Plane Layer with a Univariant Phase Transition Heated from Below

As discussed in Chapter 4, and as will be emphasized in Chapters 9 and 10, the major mantle phase transitions at depths of 410 km and 660 km have a significant influence on temperature and flow in the mantle. In (4.6.2) we discussed how phase boundary distortion and release or absorption of latent heat tend to enhance or retard material transport through a phase transition; the different effects are summarized in Table 4.6 and illustrated in Figures 4.41 and 4.42. The ability of thermal convection to occur in a fluid layer containing a phase change is strongly affected by the flow enhancement and retardation effects associated with the phase change. A linear stability analysis for the onset of thermal convection in a fluid layer with a phase change both reveals these effects and quantifies them. The basic physics of how the olivine–spinel and spinel–perovskite phase changes influence mantle convection is contained in the phase change instability problem. The onset of thermal convection in a fluid layer with a univariant phase transition heated from below has been studied by Schubert et al. (1970), Schubert and Turcotte (1971), and Busse and Schubert (1971). The linear stability problem for a fluid layer with a divariant phase change has been discussed by Schubert et al. (1975).

We consider a situation similar to that of Section 7.3 except that the fluid layer has a univariant phase change at its midpoint \( y = b/2 \). The fluid is assumed to be in thermodynamic equilibrium both in the unperturbed reference state and in the perturbed state at the onset of thermal convection. The location of the phase boundary is thus determined by the intersection of the Clapeyron curve with the pressure–temperature curve for the fluid layer. In the perturbed state the phase boundary will be distorted from its initial position
at \( y = b/2 \). The slope of the Clapeyron curve \( \Gamma \) is given by (4.6.12)

\[
\Gamma = \left( \frac{d \rho_l}{dT} \right)_c = \frac{L_H \rho_l \rho_h}{T (\rho_h - \rho_l)} \tag{7.4.1}
\]

In the cases of the olivine–spinel and spinel–perovskite + magnesiowüstite phase changes, the light phase lies above the heavy phase, which is implicit in writing (7.4.1). The olivine–spinel phase change is exothermic, \( L_H \) is positive, and \( \Gamma \) is positive. The spinel–perovskite phase change is endothermic, \( L_H \) is negative, and \( \Gamma \) is negative.

We assume that \( \rho_h - \rho_l \ll \rho_l, \rho_h \), and that both phases have the same values of dynamic viscosity \( \mu \), thermal conductivity \( k \), specific heat at constant pressure \( c_p \), thermal diffusivity \( \kappa \), and kinematic viscosity \( v \) (\( \mu, k, c_p, \kappa \), and \( v \) are constants). Each phase is assumed to be a Boussinesq fluid, i.e., the density of each phase is regarded as constant except insofar as the thermal expansion of the fluid provides a force of buoyancy. The value of the coefficient of thermal expansion \( \alpha \) is assumed to be constant and the same for both phases. The difference in density between the two phases is taken into account in determining the distortion of the phase change boundary and in the pressure boundary condition at the phase change interface. Two instability mechanisms are present in this model, the ordinary Rayleigh instability associated with the thermal expansion of the fluid and a phase change instability driven by the density difference between the phases.

In the undisturbed state there is a constant temperature gradient of magnitude \( \beta \):

\[
\beta = \frac{T_1 - T_0}{b} \tag{7.4.2}
\]

(see Section 7.3). There are also pressure gradients \( \rho_l g \) and \( \rho_h g \) in the upper and lower phases, respectively (see 7.3.5). Since the less dense phase lies above the more dense phase, \( \rho_l g/\beta > \Gamma \).

The solution to this linear stability problem can be obtained by writing equations similar to those in Section 7.3 for each of the two phases. The temperature perturbations in each of the phases \( \hat{\theta}_l^* \) and \( \hat{\theta}_h^* \) are solutions of (7.3.24) subject to the boundary conditions (7.3.25). The solutions to (7.3.24) involve 6 a priori unknown constants in each of the phases for a total of 12 unknowns to be determined as part of the solution. Boundary conditions (7.3.25) provide 6 equations for the 12 unknowns, leaving 6 unknowns still to be determined. The additional conditions follow from appropriate matching of \( \hat{\theta}_l^* \) and \( \hat{\theta}_h^* \) at the phase change interface. At the phase change boundary, mass flow, temperature, tangential stress, and tangential velocity must be continuous. In the linear stability analysis these continuity conditions can be applied at the undisturbed position \( y = b/2 \) of the phase change. Continuity of temperature is simply

\[
\theta_l^* = \theta_h^* \quad \text{at} \quad y^* = \frac{1}{2} \tag{7.4.3}
\]

Continuity of mass flux can be simplified, by using (7.3.19) and neglecting the difference between \( \rho_l \) and \( \rho_h \), to yield

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \theta_l^* = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \theta_h^* \quad \text{at} \quad y^* = \frac{1}{2} \tag{7.4.4}
\]

Continuity of tangential stress can be written, with the aid of (6.15.4), (7.3.11), and (7.3.14), in the form

\[
\frac{\partial^2}{\partial y^2} \nabla^2 \theta_l^* = \frac{\partial^2}{\partial y^2} \nabla^2 \theta_h^* \quad \text{at} \quad y^* = \frac{1}{2} \tag{7.4.5}
\]
where $\nabla^2\theta^* = \frac{\partial^2}{\partial x^*^2} + \frac{\partial^2}{\partial y^*^2}$. Continuity of tangential velocity, with the aid of (7.3.11) and (7.3.14), becomes

$$
\frac{\partial}{\partial y^*} \nabla^2 \theta^*_l = \frac{\partial}{\partial y^*} \nabla^2 \theta^*_h \quad \text{at} \quad y^* = \frac{1}{2}
$$

(7.4.6)

There are two more conditions to be satisfied at the phase change interface. One of these is a result of the mass flux across the phase boundary, which causes energy to be released or absorbed depending on the direction of the phase change. At the phase boundary an amount of energy $\rho v L_H$ is absorbed or released per unit time and per unit area. The energy absorbed or released at the interface must be balanced by the difference in the perturbation heat flux into and out of the phase boundary. This linearized condition is

$$
\frac{v^* L_H}{c_p(T_1 - T_0)} \frac{\partial \theta^*_l}{\partial y^*} - \frac{\partial \theta^*_h}{\partial y^*} \quad \text{at} \quad y^* = \frac{1}{2}
$$

(7.4.7)

or, using (7.3.14),

$$
\frac{L_H}{c_p(T_1 - T_0)} \nabla^2 \theta^* = \frac{\partial \theta^*_l}{\partial y^*} - \frac{\partial \theta^*_h}{\partial y^*} \quad \text{at} \quad y^* = \frac{1}{2}
$$

(7.4.8)

Finally, the normal stress must be continuous at the phase boundary. Because of the nonzero pressure gradient in the unperturbed state, the linearization of this condition requires introduction of the vertical displacement $\eta$ of the distorted phase boundary. Continuity of normal stress at the phase boundary can be written as

$$
\Pi^*_l - \Pi^*_h = \frac{-g b^3 (\rho_h - \rho_l)}{\mu \kappa} \eta^*
$$

(7.4.9)

In (7.4.9) the difference in perturbation pressure between the two phases is equated to the hydrostatic head generated by the density difference between the phases and the displacement of the phase boundary. This pressure difference forces the flow that can result in a phase change driven instability.

Since the condition (7.4.9) introduces an additional unknown, the phase change interface displacement, yet another equation must be used to complete the solution. This is simply the requirement that the boundary between the phases must lie on the Clapeyron curve. The phase boundary displacement can thus be related to the temperature and pressure perturbations by

$$
\eta^* = \frac{\Pi^*_l - \theta^*_l (\Gamma b^2 (T_1 - T_0)/\mu \kappa)}{(g \rho_l - \Gamma \beta)} = \frac{\Pi^*_h - \theta^*_h (\Gamma b^2 (T_1 - T_0)/\mu \kappa)}{(g \rho_h - \Gamma \beta)}
$$

(7.4.10)

The temperature and pressure perturbations in (7.4.10) are evaluated at $y^* = 1/2$. The remaining details of the solution can be found in Schubert and Turcotte (1971). Here we proceed to discuss the results.

The solution to the above system of equations depends on the following four dimensionless parameters:

$$
S = \frac{(\rho_h - \rho_l)/\rho}{\alpha b (\rho g / \Gamma - \beta)}
$$

(7.4.11)
Linear Stability

$$Ra_{LH} = \frac{\alpha gb^3 L_H/c_p}{v_k}$$

(7.4.12)

$\lambda^*$ (the dimensionless horizontal wavelength), and $Ra$ as given by (7.3.22). In writing $S$ in (7.4.11) the small difference between $\rho_h$ and $\rho_l$ has been neglected, except insofar as $S$ is proportional to $(\rho_h - \rho_l)$. The interpretation of $S$ is facilitated by rewriting (7.4.11) in the form

$$S = P \left(1 - \frac{\beta \Gamma}{\rho g}\right)^{-1}$$

(7.4.13)

where

$$P = \frac{\Gamma (\rho_h - \rho_l)}{\alpha \rho^2 gb} = \frac{(\Gamma (T_1 - T_0) / \rho g) (\rho_h - \rho_l)}{\alpha \rho b (T_1 - T_0)}$$

(7.4.14)

(see also Section 10.4). The parameter $P$ is known as the phase buoyancy parameter (Christensen and Yuen, 1985) and it is essentially identical to $S$ since the factor $(1 - \beta \Gamma/\rho g)^{-1}$ is near unity for the olivine–spinel and spinel–perovskite phase transitions. By examination of the far right side of (7.4.14) it can be seen that $P$ is the ratio of the mass per unit area due to phase boundary distortion caused by the temperature difference across the layer ($T_1 - T_0$) to the mass per unit area associated with thermal expansion and $(T_1 - T_0)$. Thus, $P$ or $S$ measures the relative importance to convection of forces associated with phase boundary distortion and thermal expansion. The parameter $Ra_{LH}$ is a Rayleigh number based on the temperature difference $L_H/c_p$. It measures the stabilizing influence of the latent heat in the olivine–spinel phase transition and the destabilizing influence of the latent heat in the spinel–perovskite phase transition (see Table 4.6). The definition of $Ra$ in (7.3.22) can be generalized to refer to the superadiabatic temperature difference across the layer, i.e., (7.3.22) can be rewritten as

$$Ra = \frac{\alpha gb^4 (T_1 - T_0)}{v_k b} = \frac{\alpha gb^4 \beta}{v_k}$$

(7.4.15)

and $\beta - \beta_{ad}$ can be substituted for $\beta$ in (7.4.15) to yield

$$Ra = \frac{\alpha gb^4 (\beta - \beta_{ad})}{v_k}$$

(7.4.16)

where $\beta_{ad}$ is the magnitude of the adiabatic temperature gradient given by (4.7.3). The generalization from (7.4.15) to (7.4.16) can be made since it is only the superadiabatic temperature difference across a fluid layer that is effective in driving convection and in the Boussinesq case of this section $\beta_{ad}$ is approximated as zero.

The limiting cases $\alpha \to 0$ and $\beta \to \beta_{ad}, \alpha \to 0$ and $\beta \neq \beta_{ad}, \alpha \neq 0$ and $\beta \to \beta_{ad}$ have been studied in detail by Busse and Schubert (1971). The results of the analysis for the case $\alpha \to 0$, $\beta \neq \beta_{ad}$ have been applied to mantle phase changes by Schubert et al. (1970). This case is of particular interest since for $\alpha = 0$ the ordinary Rayleigh instability is not present, and one may focus on the phase change as the source of instability. For the olivine–spinel phase change, the inflow of relatively cold material from above the phase change boundary (due to the zero-order temperature gradient) forces the interface to a region of lower hydrostatic pressure, i.e., upward. With the interface displaced upward, the heavier material below the interface gives a hydrostatic pressure head tending to drive the flow
downward, leading to instability. However, the downward flow of fluid through the interface releases heat, thus tending to warm the fluid and return the phase boundary to its unperturbed location. The inflow of cold material tends to promote instability, whereas release of heat by the phase change promotes stability. In regions of downward flow the phase boundary is displaced upward and in regions of upward flow the phase boundary is moved downward (see Section 4.6.2).

Figure 7.4 gives the minimum (over all possible perturbation horizontal wavelengths) critical Rayleigh number $Ra_{cr}(min)$ for the onset of thermal convection in a fluid layer with a univariant phase change at its midpoint as a function of $Ra_{LH}$ for various positive values of $S$ corresponding to an exothermic phase change like that of olivine to spinel. For $Ra_{LH} = S = 0$ there is no phase change, and the results are in agreement with those of the previous section which give $Ra_{cr}(min) = 27\pi^4/4 = 657.5$. In the limit $\alpha \to 0$, $Ra$ and $Ra_{LH} \to 0$, $S \to \infty$, and the appropriate Rayleigh number for the phase change density difference $RaS \to 327.384$ for $Ra_{LH}/Ra = 0$ (Schubert et al., 1970; Busse and Schubert, 1971). In the ordinary Rayleigh instability, the density change due to thermal expansion is spread throughout the fluid layer. In the phase change instability of a fluid with $\alpha = 0$, the density change occurs at a single position in the fluid. The Rayleigh number for the former problem can be found analytically, whereas for the latter the determination of the Rayleigh number requires the numerical evaluation of a transcendental expression.

Since, in the exothermic case, $Ra_{LH}$ represents the stabilizing effect of latent heat release at the phase change interface, it is clear from Figure 7.4 that $Ra_{cr}(min)$ increases with increasing $Ra_{LH}$ at a fixed value of $S$. As can be seen from Figure 7.4, for sufficiently large $Ra_{LH}$ the minimum critical Rayleigh number becomes insensitive to the value of $S$. Also, for sufficiently large $Ra_{LH}$, the critical Rayleigh number for symmetric convection (Figure 7.4) exceeds the critical Rayleigh number for antisymmetric convection 657.5 (Chandrasekhar, 1961). We find from Figure 7.4 that $Ra_{cr}(min)$ decreases as $S$ increases for fixed $Ra_{LH}$. This reflects the fact that as $S$ increases the fractional density change associated with the exothermic phase transition becomes increasingly significant as compared with the density change associated with thermal expansion, and the phase change plays a more important role in driving the instability, thus reducing the critical Rayleigh number. For a wide range
of values of $Ra_{L_H}$ and $S$, the critical Rayleigh number for symmetric convection through a phase change is lower than the critical Rayleigh number for a single-phase fluid.

Figure 7.5 gives the minimum critical Rayleigh number for the onset of thermal convection in a fluid layer with a univariant phase change at its midpoint as a function of $Ra_{L_H}$ for various negative values of $S$ corresponding to an endothermic phase change like that of spinel to perovskite. For the endothermic phase change both $Ra_{L_H}$ and $S$ are negative. Comparison of Figures 7.5 and 7.4 shows that the effects of $|S|$ and $|Ra_{L_H}|$ on layer instability are opposite in the endothermic case compared with the exothermic case. As $|S|$ increases in Figure 7.5, phase boundary distortion becomes more important and the layer becomes more stable. For the case $S = 0$, there is no phase boundary distortion and the latent heat effect is seen in Figure 7.5 to be a destabilizing influence (see Table 4.6).

Values of the phase change instability parameters $P$ (or $S$) and $Ra_{L_H}$ for the olivine–spinel and spinel–perovskite + magnesiowüstite phase changes can be estimated as follows. The Clapeyron slope for the olivine–spinel phase change is between 1.5 and 2.5 MPa K$^{-1}$ (Akaogi et al., 1989; Katsura and Ito, 1989) while $\Gamma$ for the spinel–perovskite phase transition is $-2$ to $-4$ MPa K$^{-1}$ (see Section 10.4). Bina and Helffrich (1994) suggest $\Gamma_{410} = 3$ MPa K$^{-1}$ and $\Gamma_{660} = -2.5$ MPa K$^{-1}$. For $(\rho H - \rho L) / \rho$ we take 0.05 for olivine–spinel and 0.010 for spinel–perovskite. We evaluate $L_H$ for both phase changes from (7.4.1) with $T = 2,000$ K. In addition we take $\alpha = 3 \times 10^{-5}$ K$^{-1}$, $\nu = 2.5 \times 10^{17}$ m$^2$s$^{-1}$, $\rho = 4,000$ kg m$^{-3}$, $g = 10$ m s$^{-2}$, $b = 10^3$ km, and $k = 4$ W m$^{-1}$ K$^{-1}$. We find $P_{440} \approx 0.1$ and $P_{660} \approx -0.25$. For the olivine–spinel transition we obtain $Ra_{L_H} \approx 6 \times 10^4$ and for the spinel–perovskite transition we get $Ra_{L_H} \approx -1.8 \times 10^5$. The maximum values of $|Ra_{L_H}|$ in Figures 7.4 and 7.5 are somewhat smaller than these estimates of $Ra_{L_H}$ for the olivine–spinel and spinel–perovskite phase changes in the mantle.
The question of whether the transition zone phase changes hinder or enhance mantle convection is best discussed using results from two-dimensional and three-dimensional numerical models of finite-amplitude thermal convection with phase changes. These results are presented in Chapters 9 and 10. The linear stability problem of this section is of value in that it brings out much of the physics that is important in how phase changes influence mantle convection.

**Question 7.1:** What is the influence of the olivine–spinel and spinel–perovskite + magnesiowüstite phase changes on mantle convection?

Peltier et al. (1989) presented a stability analysis for a spherical shell with a univariant phase transition.

### 7.5 Plane Layer Heated from Within

In this section we consider the onset of thermal convection in an infinite horizontal fluid layer heated internally and cooled from above. The entire mantle can be approximated by such a layer in that the mantle is heated primarily by the decay of radiogenic elements distributed throughout its interior; only a relatively small amount (≈10%) of the surface heat flow enters the mantle from below at the core–mantle boundary. The geometry is identical to that of Section 7.3. However, in this case the lower boundary is assumed to be insulating so that no heat flux enters the layer from below. The internal heat sources are assumed to be distributed uniformly throughout the layer with \( H = \) heat generation rate per unit mass of fluid = constant. The upper boundary is taken to be isothermal at temperature \( T_0 \). Again, either shear stress free or fixed surface boundaries can be considered. The scale for nondimensionalization of the temperature \( \Delta T_r \) is the characteristic conductive temperature difference across the fluid layer due to the internal heat production:

\[
\Delta T_r = \frac{b^2 H \rho}{k}
\]  

(7.5.1)

Length is dimensionless with respect to layer thickness \( b \) and the scale for the internal heat generation rate is \( H_r = H \) so that \( H^* = 1 \).

The dimensionless conduction temperature in this case is a solution of (7.2.10) subject to the above boundary conditions:

\[
T_c^* = \frac{T_0}{(b^2 H \rho / k)} + y^* - \frac{y^{*2}}{2}
\]  

(7.5.2)

The nondimensional conductive heat flux in the vertically downward direction \( q_c^* \) is given by

\[
q_c^* = \frac{q_c}{b H \rho} = \frac{k (dT_c/dy)}{b H \rho} = \frac{dT_c^*}{dy^*}
\]  

(7.5.3)

From the above two expressions we can write

\[
q_c^* = -(1 - y^*)
\]  

(7.5.4)
At the bottom of the layer, \( y^* = 1 \), \( q_c^* \) is zero, while at the top, \( y^* = 0 \), \( q_c^* \) is \(-1\); in between, \( q_c^* \) varies linearly with depth. The conductive heat flux is negative since heat flows upward. A simple energy balance requires that at any position in the layer, the upward conductive heat flux is the product of the volumetric heat production rate \( \rho H \) and the thickness of the underlying layer \((b - y)\).

The pressure in the motionless conductive basic state is the sum of \( \overline{p} \) given by (7.3.6) and \( p_c \), the hydrostatic pressure associated with the variation in \( T_c \) with \( y \). The equation for \( p_c^* = p_c \left[(\mu_r/b)(k_r/\rho_r c_p r b)\right]^{-1} \) is (7.2.5) or

\[
\frac{dp_c^*}{dy^*} = -Ra_H T_c^*
\]  

(7.5.5)

where \( Ra_H \) is the internal heating Rayleigh number which, from (6.10.20) and (7.5.1), is

\[
Ra_H = \frac{\alpha g \rho H b^5}{k \kappa v}
\]  

(7.5.6)

With \( T_c^* \) given by (7.5.2), (7.5.5) becomes

\[
\frac{dp_c^*}{dy^*} = -Ra_H \left\{ \frac{T_0}{b^2 H \rho / k} + y^* - \frac{y^{*2}}{2} \right\}
\]  

(7.5.7)

which integrates to

\[
p_c^* = -Ra_H \left\{ \frac{T_0}{b^2 H \rho / k} y^* + \frac{y^{*2}}{2} - \frac{y^{*3}}{6} \right\}
\]  

(7.5.8)

(the constant of integration is zero). The dimensional form of (7.5.8) is

\[
p_c = -\alpha \rho g \left\{ T_0 y + \frac{1}{2} \left( \frac{\rho H b}{k} \right) y^2 - \frac{1}{6} \left( \frac{\rho H}{k} \right) y^3 \right\}
\]  

(7.5.9)

The equations for the dynamically induced perturbations \( u^* \), \( v^* \), \( \Pi^* \), and \( \theta^* \) are (7.3.11)–(7.3.13) and (7.2.11) which, from (7.5.3), can be written as

\[
(1 - y^*)v^* = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^*^2} \right) \theta^*
\]  

(7.5.10)

As was the case for the stability of the plane layer heated from below, we limit ourselves here to the two-dimensional stability of the internally heated plane layer.

For shear stress free and impermeable surfaces the velocity boundary conditions are

\[
v^* = \frac{\partial u^*}{\partial y^*} = 0 \quad \text{on} \quad y^* = 0, 1
\]  

(7.5.11)

while for rigid and impermeable surfaces the velocity boundary conditions are

\[
v^* = u^* = 0 \quad \text{on} \quad y^* = 0, 1
\]  

(7.5.12)

With the upper surface isothermal and the lower surface insulating, the temperature boundary conditions are

\[
\theta^* = 0 \quad \text{on} \quad y^* = 0 \quad \text{and} \quad \frac{\partial \theta^*}{\partial y^*} = 0 \quad \text{on} \quad y^* = 1
\]  

(7.5.13)
The solution to this stability problem can be obtained using the approach of Section 7.3 (7.3.17–7.3.21). Equation (7.3.18) is still valid (with \( Ra \) replaced by \( Ra_H \)), while (7.3.19) is replaced by (7.5.10) or

\[
0 = \nabla^2 \psi^* + \frac{\partial \psi^*}{\partial x^*}(1 - y^*) \quad (7.5.14)
\]

By differentiating (7.5.14) with respect to \( x^* \), taking the Laplacian of (7.3.18), and combining the two results, we obtain a single partial differential equation for \( \psi^* \):

\[
0 = \nabla^6 \psi^* - Ra_H (1 - y^*) \frac{\partial^2 \psi^*}{\partial x^*} \quad (7.5.15)
\]

For shear stress free surfaces the boundary conditions in terms of \( \psi^* \) follow from (7.3.17), (7.3.18), (7.5.11), and (7.5.13):

\[
\frac{\partial \psi^*}{\partial x^*} = \frac{\partial^2 \psi^*}{\partial y^*} = 0 \quad \text{on} \quad y^* = 0, 1 \quad (7.5.16)
\]

\[
\frac{\partial^4 \psi^*}{\partial y^*} = 0 \quad \text{on} \quad y^* = 0 \quad \text{and} \quad \nabla^4 \frac{\partial \psi^*}{\partial y^*} = 0 \quad \text{on} \quad y^* = 1 \quad (7.5.17)
\]

For rigid surfaces, the conditions on \( \frac{\partial^2 \psi^*}{\partial y^*} \) in (7.5.16) are replaced by

\[
\frac{\partial \psi^*}{\partial y^*} = 0 \quad \text{on} \quad y^* = 0, 1 \quad (7.5.18)
\]

and (7.5.17) is replaced by

\[
\nabla^4 \psi^* = 0 \quad \text{on} \quad y^* = 0, \quad \nabla^4 \frac{\partial \psi^*}{\partial y^*} = 0 \quad \text{on} \quad y^* = 1 \quad (7.5.19)
\]

As in the preceding section, we can assume \( \psi^* \) is periodic in the horizontal coordinate \( x^* \):

\[
\psi^* = \hat{\psi}^*(y^*) \sin \frac{2\pi x^*}{\lambda^*} \quad (7.5.20)
\]

where \( \lambda^* \) is the a priori unknown dimensionless horizontal wavelength of the perturbation stream function. Substitution of (7.5.20) into (7.5.15)–(7.5.19) yields

\[
\left( \frac{d^2}{dy^*} - \frac{4\pi^2}{\lambda^*} \right)^3 \hat{\psi}^*(y^*) = -Ra_H (1 - y^*) \frac{4\pi^2}{\lambda^*} \hat{\psi}^*(y^*) \quad (7.5.21)
\]

\[
\hat{\psi}^* = \frac{d^2 \hat{\psi}^*}{dy^*} = 0 \quad \text{on} \quad y^* = 0, 1 \quad \text{(shear stress free surfaces)} \quad (7.5.22)
\]
\[
\frac{d^4 \hat{\psi}^*}{d y^4} = 0 \quad \text{on} \quad y^* = 0 \quad \text{(shear stress free surface)}
\]
\[
\left( \frac{d^2}{d y^2} - \frac{4 \pi^2}{\lambda^*} \right)^2 \frac{d \hat{\psi}^*}{d y^*} = 0 \quad \text{on} \quad y^* = 1 \quad \text{(shear stress free surface)}
\] (7.5.23)
\[
\hat{\psi}^* = \frac{d \hat{\psi}^*}{d y^*} = 0 \quad \text{on} \quad y^* = 0, 1 \quad \text{(rigid surfaces)}
\] (7.5.24)
\[
\left( \frac{d^2}{d y^2} - \frac{4 \pi^2}{\lambda^*} \right)^2 \hat{\psi}^* = 0 \quad \text{on} \quad y^* = 0 \quad \text{(rigid surface)}
\] (7.5.25)
\[
\left( \frac{d^2}{d y^2} - \frac{4 \pi^2}{\lambda^*} \right)^2 \frac{d \hat{\psi}^*}{d y^*} = 0 \quad \text{on} \quad y^* = 1 \quad \text{(rigid surface)}
\]

A simple analytic solution of (7.5.21)–(7.5.25) is not possible because (7.5.21) is an ordinary differential equation with nonconstant coefficients. The analogous equation for the heated from below stability problem (7.3.24) is a constant coefficient ordinary differential equation. The source of the difficulty is the basic state conductive temperature gradient which is constant when heating is from below and a linear function of depth when heating is from within. Nevertheless, a rather straightforward numerical solution of this system of equations is obtainable by standard techniques.

A solution of this problem has been obtained by Roberts (1967). With an insulated (zero heat flux) lower boundary, an isothermal upper boundary, and free-surface boundary conditions the minimum critical value of this Rayleigh number is \( Ra_{H, cr} \text{(min)} = 867.8 \) and the corresponding nondimensional wavelength is \( \lambda_{cr}^* = 3.51 \). For fixed surface boundary conditions \( Ra_{H, cr} \text{(min)} = 2.772 \) and \( \lambda_{cr}^* = 2.39 \) and for a free upper boundary and fixed lower boundary \( Ra_{H, cr} \text{(min)} = 1.612.6 \) and \( \lambda_{cr}^* = 2.78 \).

It is appropriate to apply the Rayleigh number based on internal heat generation to whole mantle convection. This neglects the flux of heat out of the core but this is estimated to be only about 10% of the surface heat flux. The heat generation in the mantle \( H \) can be estimated by dividing the total mantle heat flow \( Q_m = 3.69 \times 10^{13} \text{ W} \) obtained above by the mass of the mantle \( M_m = 4.0 \times 10^{24} \text{ kg} \); the result is \( H = 9.23 \times 10^{-12} \text{ W kg}^{-1} \). It will be shown in Chapter 13 that part of the equivalent heat generation is due to the secular cooling of the mantle and part is due to heat produced by the radioactive isotopes of uranium, thorium, and potassium.

In order to complete the specification of the Rayleigh number for whole mantle convection, we take \( \rho = 4.6 \times 10^3 \text{ kg m}^{-3} \), \( b = 2.900 \text{ km} \), and the other values used above for layered upper mantle convection. Substitution of these values into (7.5.6) gives \( Ra_H = 3 \times 10^9 \). Thus the ratio of the Rayleigh number \( Ra_H \) to the minimum critical Rayleigh number for whole mantle convection \( Ra_{H, cr} \text{(min)} \) is between \( r = Ra_H / Ra_{H, cr} \text{(min)} = 3.6 \times 10^6 \) and \( 1.2 \times 10^6 \) depending upon the applicable boundary conditions. A comparison of this result with that obtained in Section 7.3 shows that the value of \( r \) is about a factor of 100 larger for whole mantle convection than it is for layered mantle convection. The requirement that the Rayleigh number be greater than \( 10^9 \) for simulations of whole mantle convection provides an important constraint on numerical models of mantle convection, as we will discuss in later chapters.
7.6 Semi-infinite Fluid with Depth-dependent Viscosity

We next consider the onset of thermal convection in a semi-infinite fluid with a strongly depth-dependent viscosity. In the linear stability problem a depth-dependent viscosity can take account of any pressure and temperature dependence that the viscosity may have since the required viscosity is only a function of the zero-order temperature and pressure that in turn are only functions of depth. If it is assumed that the viscosity has an exponential depth dependence given by

$$\mu = \mu_s e^{y/h}$$  \hspace{1cm} (7.6.1)

where \(\mu_s\) is the surface viscosity at \(y = 0\), \(y\) is measured downward, and \(h\) is the scale depth of the viscosity variation, then the appropriate Rayleigh number is

$$Ra_\mu = \frac{\rho g \alpha \beta h^4}{\kappa \mu_s}$$  \hspace{1cm} (7.6.2)

where \(\beta\) is the temperature gradient in the semi-infinite fluid. The solution to this problem has been obtained by Schubert et al. (1969). For a free surface boundary condition \(Ra_{\mu, cr}(\text{min}) = 23\) and \(\lambda_{cr}^* = 13.1\), and for a fixed surface boundary condition \(Ra_{\mu, cr}(\text{min}) = 30\) and \(\lambda_{cr}^* = 12.3\). The critical Rayleigh numbers are small and the critical nondimensional wavelengths are large because the convective flow extends several scale depths into the fluid. The effects of small variations in fluid properties on the stability problem have been studied by Palm (1960), Segel and Stuart (1962), Palm and Øiann (1964), Segel (1965), Busse (1967), and Palm et al. (1967). These studies were primarily concerned with the form and direction of the flow, i.e., whether the flow takes the form of two-dimensional rolls or three-dimensional hexagons. McKenzie (1988) has examined the symmetries of convective transitions in space and time.

A related problem is the stability of a thickening thermal boundary layer in a semi-infinite fluid. If the semi-infinite fluid initially has a uniform temperature \(T\), and the temperature of the upper surface is instantaneously reduced to \(T_0\), a thermal boundary layer develops. Its structure has been given in (4.1.20). In the presence of a gravitational field the cold, dense thermal boundary layer is unstable. The tendency of the dense thermal boundary layer to sink into the underlying less dense fluid can be analyzed as a linear stability problem for the onset of thermal convection. The thermal boundary layer thickens with time until it reaches a critical thickness when convection begins. The boundary layer thickness \(y_T\) has been defined in (4.1.22). The applicable Rayleigh number for this problem is defined in terms of the thickness of the thermal boundary layer rather than the thickness of the full layer:

$$Ra_{y_T} = \frac{\alpha g (T_1 - T_0) y_T^3}{\nu \kappa}$$  \hspace{1cm} (7.6.3)

For free surface boundary conditions Jaupart and Parsons (1985) found that the minimum critical value of this Rayleigh number is \(Ra_{y_T, cr}(\text{min}) = 807\). Substitution of this value into (7.6.3) yields a formula for the thickness of the boundary layer at the onset of convection:

$$y_T = \left\{ \frac{807 \nu \kappa}{\alpha g (T_1 - T_0)} \right\}^{1/3}$$  \hspace{1cm} (7.6.4)
With \( \mu = 10^{21} \text{Pa}s, \rho = 3,300 \text{kg m}^{-3}, \kappa = 1 \text{mm}^2 \text{s}^{-1}, \alpha = 3 \times 10^{-5} \text{K}^{-1}, g = 10 \text{m s}^{-2}, \) and \( T_1 - T_0 = 1,300 \text{K}, \) (7.6.4) gives \( y_T = 85.6 \text{km} \) corresponding, according to (4.2.8), to an age of 43.3 Myr. According to this constant viscosity theory, one would expect the onset of secondary convection beneath the oceanic lithosphere to set in at an age of about 43 Myr. In Chapter 8 we will utilize this boundary layer instability to derive approximate solutions for fully developed thermal convection.

Jaupart and Parsons (1985) have also considered this problem for a depth-dependent viscosity of the form

\[
\mu = \mu_1 + (\mu_0 - \mu_1) e^{-y/h} \tag{7.6.5}
\]

where \( \mu_0 \) is the value of the viscosity at the surface \( y = 0 \) and \( \mu_1 \) is the value of the viscosity at great depth \( y \to \infty \). They found a bifurcation in the stability problem for intermediate values of the viscosity contrast \( \mu_0/\mu_1 \). For \( \mu_0/\mu_1 < 10^3 \), the instability includes the entire thermal boundary layer, as in the constant viscosity case. For \( \mu_0/\mu_1 > 10^3 \) only the lower part of the thermal boundary layer becomes unstable and a “stagnant-lid” develops (see the discussion of stagnant-lid convection in Chapters 13 and 14). The whole-layer instability is associated with subduction while the partial layer instability is associated with delamination or secondary convection. Laboratory experiments carried out by Davaille and Jaupart (1994) have verified this bifurcation. These authors concluded that the asthenosphere viscosity must be in the range \( \mu = 3 \times 10^{18} \) to \( 4 \times 10^{17} \text{Pa}s \) for secondary convection to develop beneath the oceanic lithosphere.

Yuen et al. (1981) also considered this problem for a depth-dependent viscosity. In their model, the dependence of viscosity on depth arose from the strong dependence of \( \mu \) on temperature and the dependence of temperature on depth appropriate to the oceanic lithosphere given by (4.2.4). They also concluded that secondary convection beneath the oceanic lithosphere required the existence of a pronounced low-viscosity asthenosphere beneath the lithosphere.

In all the above linear stability problems wherein viscosity varies with depth, the relevant equation for the perturbation stream function can be easily obtained from (6.11.1) and (6.11.2). When \( \mu = \mu(y) \), (6.11.1) can be written (with \( \zeta \) eliminated using (6.11.2)) as

\[
\nabla^4 \psi + \frac{2}{\mu} \frac{d\mu}{dy} \frac{\partial}{\partial y} \nabla^2 \psi = -\frac{\alpha \rho g}{\mu} \frac{\partial T'}{\partial x} + \frac{1}{\mu} \frac{d^2 \mu}{dy^2} (\psi_{xx} - \psi_{yy}) \tag{7.6.6}
\]

**Question 7.3:** Is there secondary or small-scale thermal convection in the oceanic asthenosphere due to gravitational instability of the lower part of the oceanic lithosphere?

### 7.7 Fluid Spheres and Spherical Shells

In this section we consider the onset of thermal convection in homogeneous fluid spheres and spherical shells. With the application to planets in mind, we take the force of gravity to be radially inward and let \( \bar{g} = \bar{g}(r) \), where \( r \) is the radial coordinate. We adopt the Boussinesq approximation and consider \( \bar{\rho} = \text{constant} \). The reference state for these models was discussed in Section 6.12. From (6.12.5) and (6.12.6) we can write \( \bar{g}(r) \) and \( \bar{\rho}(r) \) for a
sphere of radius \( a \) as

\[
\overline{g}(r) = \frac{4}{3} \pi G r \bar{\rho} \quad (7.7.1)
\]

\[
\overline{p}(r) = \frac{2}{3} \pi G \bar{p}^2 (a^2 - r^2) \quad (7.7.2)
\]

For a spherical shell of inner radius \( c \), outer radius \( a \), and constant density \( \bar{\rho} \) surrounding a core of average density \( \rho_c \), \( \overline{g}(r) \) and \( \overline{p}(r) \) are given by (6.12.2) and (6.12.4)

\[
\overline{g}(r) = \frac{4}{3} \pi G \left\{ r \bar{\rho} + \frac{c^3}{r^2} (\rho_c - \bar{\rho}) \right\}, \quad c \leq r \leq a \quad (7.7.3)
\]

\[
\overline{p}(r) = \frac{4}{3} \pi \bar{\rho} G c^3 (\rho_c - \bar{\rho}) \left( \frac{1}{r} - \frac{1}{a} \right) + \frac{2}{3} \pi G \bar{p}^2 (a^2 - r^2), \quad c \leq r \leq a \quad (7.7.4)
\]

and \( \overline{p}(a) \) has been taken as zero.

The temperature in the reference state \( T_c \) is the conduction solution given by (6.12.11). For the uniformly heated sphere the constant \( c_1 \) is zero, and \( c_2 \) can be determined from the condition \( T_c(r = a) = 0 \) with the result

\[
T_c = -\frac{\overline{p} H}{6k} (r^2 - a^2) \quad (7.7.5)
\]

For the spherical shell with isothermal inner and outer boundaries maintained at temperatures \( T_c(r = c) = T_1 \) and \( T_c(r = a) = 0 \), (6.12.11) gives

\[
T_c = -\frac{\overline{p} H}{6k} r^2 + \frac{1}{r} \left[ \frac{T_1 - (\overline{p} H/6k)(a^2 - c^2)}{(1/c - 1/a)} \right] + \frac{\overline{p} H}{6k} (a^2 + c^2 + ac)
\]

\[-\frac{T_1 c}{(a - c)}, \quad c \leq r \leq a \quad (7.7.6)
\]

For the sphere, the radial temperature gradient \( \beta_c \) in the conductive reference state is

\[
\beta_c(r) = \frac{dT_c}{dr} = -\frac{\overline{p} H}{3k} r \quad (7.7.7)
\]

while for the spherical shell \( \beta_c \) is

\[
\beta_c = -\frac{\overline{p} H}{3k} r - \frac{1}{r^2} \left[ \frac{T_1 - (\overline{p} H/6k)(a^2 - c^2)}{(1/c - 1/a)} \right], \quad c \leq r \leq a \quad (7.7.8)
\]

The heat flux in the conductive reference state \( q_c \) is given by Fourier's law in the form

\[
q_c = -k \beta_c \quad (7.7.9)
\]

Clearly \( q_c \) is a function of \( r \). At the outer surface \( r = a \), \( q_c \) for the sphere is

\[
q_c(r = a) = -k \beta_c(r = a) = \frac{\overline{p} H a}{3} \quad (7.7.10)
\]

and \( q_c \) for the spherical shell is

\[
q_c(r = a) = -k \beta_c(r = a) = \frac{\overline{p} H a}{3} + \frac{1}{a^2} \left[ kT_1 - \frac{(\overline{p} H/6)(a^2 - c^2)}{(1/c - 1/a)} \right]
\]

\[= \frac{\overline{p} H a}{3} - \frac{\overline{p} H a c}{6} (a + c) + \frac{kT_1}{a^2} \quad (7.7.11)
\]

The spherical shell is heated partly from within and partly from below.
A complete specification of the reference state requires determination of the hydrostatic pressure \( p_c \) due to the variation in \( T_c \). From (6.10.24) we can write

\[
\frac{dp_c}{dr} = \bar{\rho} \bar{g}(r)\bar{\alpha}T_c(r) \tag{7.7.12}
\]

In this Boussinesq model we treat the thermal expansivity \( \bar{\alpha} \) as constant. Equation (7.7.12) can be integrated to give \( p_c(r) \) by using the above expressions for \( \bar{g}(r) \) and \( T_c(r) \). For the uniformly heated sphere we obtain

\[
p_c = -\frac{2\pi G}{9k} \bar{\alpha} \bar{\rho}^3 H \left( \frac{r^4}{4} - \frac{a^2r^2}{2} \right) \tag{7.7.13}
\]

(the constant of integration is zero). The total pressure in the motionless basic state for the sphere is the sum of (7.7.2) and (7.7.13):

\[
p = \frac{2}{3} \pi G \bar{\rho}^2 \left\{ a^2 - r^2 - \frac{\bar{\alpha} \bar{\rho} H}{3k} \left( \frac{r^4}{4} - \frac{a^2r^2}{2} \right) \right\} \tag{7.7.14}
\]

For the spherical shell, integration of (7.7.12) gives

\[
p_c = \frac{4}{3} \pi G \bar{\rho}^2 \left[ -\frac{\bar{\rho}H}{6k} \left\{ \frac{r^4}{4} + \frac{rc^3 \left( \rho_c - \bar{\rho} \right)}{\bar{\rho}} \right\} + \frac{T_1 - \left( \bar{\rho}H/6k \right) \left( a^2 - c^2 \right)}{(1/c - 1/a)} \right]
\]

\[\times \left\{ r - \frac{c^3 \left( \rho_c - \bar{\rho} \right)}{\bar{\rho}} \right\} + \frac{\bar{\rho}H}{6k} \left( a^2 + c^2 + ac - \frac{T_1c}{a - c} \right)\right]\]

\[\times \left\{ \frac{r^2}{2} - \frac{c^3 \left( \rho_c - \bar{\rho} \right)}{r} \right\}, \quad c \leq r \leq a \tag{7.7.15}
\]

The total pressure in the motionless basic state for the spherical shell is the sum of (7.7.4) and (7.7.15).

The linearized equations for the dynamically induced infinitesimal perturbations at convection onset \( u' \), \( \Pi' \), and \( \theta' \) are the dimensional versions of (7.2.7), (7.2.8), and (7.2.11):

\[
\nabla \cdot u' = 0 \tag{7.7.16}
\]

\[
0 = -\nabla \Pi' + \bar{\alpha} \bar{\rho} \bar{g}(r)\hat{r}\theta' + \mu \nabla^2 u' \tag{7.7.17}
\]

\[
u' \cdot \nabla T_c = \kappa \nabla^2 \theta' \tag{7.7.18}
\]

where \( \hat{r} \) is the radial unit vector. The dynamically induced pressure perturbation \( \Pi' \) can be eliminated from the equations by taking the curl of (7.7.17) and using \( \nabla \times \hat{r} = 0 \):

\[
0 = \bar{\alpha} \bar{\rho} \bar{g}(r) \nabla \theta' \times \hat{r} + \mu \nabla^2 \omega' \tag{7.7.19}
\]

where \( \omega' \) is the perturbation vorticity \( \omega' \equiv \nabla \times u' \). The perturbation temperature equation (7.7.18) can be simplified by noting that \( \nabla T_c = \beta_c(r)\hat{r} \hat{r} \):

\[
u' \beta_c(r) = \kappa \nabla^2 \theta' \tag{7.7.20}
\]
In Section 7.8 dealing with spherical harmonics, we discuss how any solenoidal vector field (a vector field with zero divergence) can be represented by the sum of poloidal and toroidal vector fields. The velocity field \( u' \) is a solenoidal vector field by virtue of (7.7.16) and it is convenient for the solution of the present problem to represent \( u' \) accordingly:

\[
    u' = \nabla \times (\nabla \times (\Phi \hat{r})) + \nabla \times (\Psi \hat{r}) \tag{7.7.21}
\]

where \( \Phi \) is the poloidal scalar and \( \Psi \) is the toroidal scalar (see Section 7.8). This representation of the perturbation velocity field automatically satisfies the continuity equation (7.7.16). The components of the perturbation velocity and vorticity fields can be written in terms of \( \Phi \) and \( \Psi \) as

\[
    u'_r = \frac{1}{r^2} L^2 \Phi \tag{7.7.22}
\]

\[
    u'_\theta = \frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial \phi} \tag{7.7.23}
\]

\[
    u'_\phi = \frac{1}{r \sin \theta} \frac{\partial^2 \Phi}{\partial r \partial \phi} - \frac{1}{r} \frac{\partial \Psi}{\partial \theta} \tag{7.7.24}
\]

\[
    \omega'_r = \frac{1}{r^2} L^2 \Psi \tag{7.7.25}
\]

\[
    \omega'_\theta = \frac{1}{r} \frac{\partial \Psi}{\partial r} - \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left\{ \nabla^2 \left( \frac{\Phi}{r} \right) \right\} \tag{7.7.26}
\]

\[
    \omega'_\phi = \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r} + \frac{\partial}{\partial \theta} \left\{ \nabla^2 \left( \frac{\Phi}{r} \right) \right\} \tag{7.7.27}
\]

where \((r, \theta, \phi)\) are spherical coordinates, and \( L \) is a horizontal differential operator defined in the next section.

Only the poloidal function \( \Phi \) contributes to the radial component of the velocity while only the toroidal function \( \Psi \) contributes to the radial component of the vorticity. As a result, the temperature equation (7.7.20) does not involve the toroidal function \( \Psi \). Inspection of the vorticity equation (7.7.19) shows that its radial component is independent of \( \theta' \) and \( \Phi \). Therefore, the toroidal function \( \Psi \) is identically zero in this onset of convection problem, i.e., buoyancy forces only induce a poloidal velocity field through the coupling of \( \theta' \) and \( \Phi \) in (7.7.20) and in the horizontal components of (7.7.19).

The problem of onset of convective instability in the sphere or spherical shell reduces to the solution of (7.7.20) and (7.7.19) for \( \theta' \) and \( \Phi \). The solution can be most readily obtained by an expansion of \( \theta' \) and \( \Phi \) in terms of spherical harmonics (see Section 7.8 for a discussion of these functions):

\[
    \theta' = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \theta'_l m(r) Y^m_l (\theta, \phi) \tag{7.7.28}
\]
\[ \Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi_{lm}(r) Y_l^m(\theta, \phi) \]  

(7.7.29)

Substitution of (7.7.28) and (7.7.29) into (7.7.19) and (7.7.20) results in

\[ \frac{\bar{\alpha} \bar{\rho} \bar{g}(r)}{\mu r} \theta_{lm}' = D_l^2 \left( \frac{\Phi_{lm}(r)}{r} \right) \]  

(7.7.30)

\[ \frac{\beta_c(r) l(l+1)}{r^2} \Phi_{lm}(r) = \kappa D_l \left( \theta_{lm}'(r) \right) \]  

(7.7.31)

where \( D_l \) is the differential operator given in (7.8.17); the \( \theta \) and \( \phi \) components of (7.7.19) are redundant, both leading to (7.7.30).

Equations (7.7.30) and (7.7.31) are coupled ordinary differential equations in \( r \) for the spherical harmonic components of the perturbation temperature field and the perturbation poloidal velocity field at the onset of convection. A single differential equation for the dynamically induced temperature perturbation \( \theta_{lm}'(r) \) can be obtained by solving (7.7.31) for \( \Phi_{lm}(r)/r \) and substituting in (7.7.30) with the result

\[ \frac{l(l+1)\bar{\alpha} \bar{\rho} \bar{g}(r)}{\kappa \mu r} \theta_{lm}'(r) = D_l^2 \left\{ \frac{r}{\beta_c(r)} D_l \left( \theta_{lm}'(r) \right) \right\} \]  

(7.7.32)

For the internally heated sphere, \( \bar{g}(r) \) is given by (7.7.1) and \( \beta_c(r) \) is given by (7.7.7); substitution of these formulæ into (7.7.32) gives

\[ D_l^{*3} \left( \theta_{lm}' \right) = -\frac{4\pi}{9} l(l+1) Ra_{sp} \theta_{lm}' \]  

(7.7.33)

where \( D_l^* \) is the dimensionless form of (7.8.17) with the radius of the sphere as the length scale, and \( Ra_{sp} \) is the appropriate Rayleigh number for the internally heated sphere given by

\[ Ra_{sp} \equiv \frac{\bar{\alpha} \bar{\rho}^3 H \sigma^6}{k \kappa \mu} \]  

(7.7.34)

For the spherical shell, \( \bar{g}(r) \) is given by (7.7.3) and \( \beta_c(r) \) is given by (7.7.8); substitution of these formulæ into (7.7.32) yields

\[ D_l^{*2} \left\{ \left[ 1 + \frac{(a^3/2r^3) \left( 6kT_1/\bar{\rho} Ha^2 - (1 - (c^2/a^2)) \right)}{(a/c - 1)} \right]^{-1} D_l^* \left( \theta_{lm}' \right) \right\} \]
\[ = -\frac{4\pi}{9} l(l+1) Ra_{sp} \left\{ 1 + \frac{c^3}{r^3} \left( \frac{\rho_c}{\bar{\rho}} - 1 \right) \right\} \theta_{lm}' \]  

(7.7.35)

Equations (7.7.33) and (7.7.35) are solved subject to boundary conditions appropriate to isothermal, impermeable, and either rigid (no-slip) or shear stress free boundary conditions. For the internally heated sphere, these conditions are applied only at the outer boundary, with the additional requirement that the solution remains finite at the origin \( r = 0 \). Impermeability of a spherical surface requires \( u_r = 0 \) on the surface. From (7.7.22), (7.7.28), and \( L^2 Y_l^m = l(l+1) Y_l^m \), we can write the impermeability condition for each spherical
harmonic as $\Phi_{lm} = 0$ on $r = a$ and on $r = c$ for the spherical shell. The condition of isothermality on the spherical surfaces is, from (7.7.28), $\theta'_{lm} = 0$ on $r = a$ and on $r = c$ for the shell. Equation (7.7.31) shows that the condition $\Phi_{lm} = 0$ is equivalent to $D_t \theta'_{lm} = 0$. If a bounding spherical surface is rigid, then the tangential velocity components $u'_\theta$ and $u'_\phi$ must vanish on the surface. From (7.7.23), (7.7.24), and (7.7.29) it is seen that $u'_\theta = u'_\phi = 0$ is equivalent to $d\Phi_{lm}/dr = 0$.

If a bounding spherical surface is shear stress free, then the viscous stresses $\tau'_{r\theta}$ and $\tau'_{r\phi}$ must vanish on the surface. The viscous shear stresses are given by (6.15.17) and (6.15.18); vanishing of these stresses on a spherical surface requires

$$
\frac{r}{r} \frac{\partial}{\partial r} \left( \frac{u'_\theta}{r} \right) + \frac{1}{r} \frac{\partial u'_r}{\partial \theta} = 0
$$

(7.7.36)

and

$$
\frac{1}{r \sin \theta} \frac{\partial u'_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{u'_\phi}{r} \right) = 0
$$

(7.7.37)

on the surface. Since $u'_r = 0$ on the surface, $\partial u'_r / \partial \theta$ and $\partial u'_r / \partial \phi$ are also zero on the surface and (7.7.36) and (7.7.37) simplify to

$$
\frac{\partial}{\partial r} \left( \frac{u'_\theta}{r} \right) = \frac{\partial u'_\theta}{\partial r} - \frac{1}{r} u'_\theta = 0
$$

(7.7.38)

$$
\frac{\partial}{\partial r} \left( \frac{u'_\phi}{r} \right) = \frac{\partial u'_\phi}{\partial r} - \frac{1}{r} u'_\phi = 0
$$

(7.7.39)

on the spherical surface. From (7.7.23) and (7.7.24), these conditions are equivalent to

$$
\frac{d}{dr} \left\{ 1 \frac{d \Phi_{lm}}{dr} \right\} = 0
$$

(7.7.40)

on the surface; since $\Phi_{lm} = 0$ on the spherical surface (7.7.40) is also equivalent to

$$
\frac{d^2}{dr^2} \left( \frac{\Phi_{lm}}{r} \right) = 0
$$

(7.7.41)

on a shear stress free spherical surface.

### 7.7.1 The Internally Heated Sphere

The onset of thermal convection in a uniformly heated sphere occurs at a value of $Ra_s p$ determined by the solution of (7.7.33) subject to the conditions

$$
\theta'_{lm} = D_t^s \theta'_{lm} = 0 \quad \text{on} \quad \frac{r}{a} = 1
$$

(7.7.42)

and either

$$
\frac{d}{d(r/a)} D_t^s \theta'_{lm} = 0 \quad \text{on} \quad \frac{r}{a} = 1
$$

(7.7.43)
for a rigid surface or
\[
\frac{d^2}{d(r/a)^2} D_l^* \theta_{lm} = 0 \quad \text{on} \quad \frac{r}{a} = 1
\]  
(7.7.44)

for a shear stress free surface. Chandrasekhar (1961) proves the validity of the exchange of stabilities for this problem, an assumption we adopted at the beginning of our analysis with the neglect of the $\partial \theta^l / \partial t$ term in (7.7.18). Though there is no simple analytic solution to this problem, a numerical solution of the ordinary differential equation (7.7.33) subject to (7.7.42) and either (7.7.43) or (7.7.44) is straightforward. The resulting values of $Ra_{sp}$, denoted by $Ra_{sp, cr}$, are given in Table 7.2 for both the rigid and shear stress free boundary conditions on the surface of the sphere. The values of $Ra_{sp, cr}$ in the table are from Chandrasekhar (1961); solutions of this linearized stability problem have also been obtained by Jeffreys and Bland (1951), Chandrasekhar (1952, 1953), Backus (1955), Roberts (1965a), and Zebib et al. (1983).

The critical Rayleigh numbers depend only on the value of $l$, i.e., for a given $l$, all convective modes with $-l \leq m \leq l$ have the same value of $Ra_{sp, cr}$ and are equally likely to occur at the onset of convection, according to linear theory. A nonlinear theory is required to determine the modes with preferred values of $m$ when $Ra_{sp} > Ra_{sp, cr}$. Convective modes with a given value of $l$ cannot occur unless $Ra_{sp} \geq Ra_{sp, cr}(l)$; when the equality holds $m$ can have any value between $-l$ and $l$, when the inequality holds there are preferred values of $m$ that can be determined from a nonlinear analysis of the convective problem. As Table 7.2 shows, the minimum critical Rayleigh number for the free surface boundary condition is $Ra_{sp, cr}(\text{min}) = 2.2139 \times 10^3$ and for the fixed surface boundary condition it is $Ra_{sp, cr}(\text{min}) = 5.7583 \times 10^3$. In both cases the minimum critical Rayleigh number is associated with the $l = 1$ mode, a single convective cell. The pattern of the axisymmetric $l = 1$ mode of convection is shown by the isotherms and streamlines in a meridional cross-section in Figure 7.6. The value of $Ra_{sp}$ for the case shown in the figure is 9.550, so convection is somewhat nonlinear and spherical harmonic modes with $l \neq 1$ are also present. Nevertheless, the flow is essentially similar to convection at the onset of instability. There is upwelling at one pole and downwelling at the other. The highest velocities occur near the outer boundary as fluid crosses the equatorial plane.

It is of interest to apply the above results to the Moon. We assume that the Moon has a rigid outer shell with a thickness of 300 km and therefore take $a = 1.400 \text{ km}$.

<table>
<thead>
<tr>
<th>$l$</th>
<th>Shear Stress Free Outer Boundary</th>
<th>Rigid Outer Boundary</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Ra_{sp, cr}$</td>
<td>$Ra_{sp, cr}$</td>
</tr>
<tr>
<td>1</td>
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<td>$5.7583 \times 10^3$</td>
</tr>
<tr>
<td>2</td>
<td>$3.7415 \times 10^3$</td>
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<tr>
<td>3</td>
<td>$6.2843 \times 10^3$</td>
<td>$1.0818 \times 10^4$</td>
</tr>
<tr>
<td>4</td>
<td>$1.0013 \times 10^4$</td>
<td>$1.5727 \times 10^4$</td>
</tr>
<tr>
<td>5</td>
<td>$1.5186 \times 10^4$</td>
<td>$2.2373 \times 10^4$</td>
</tr>
<tr>
<td>6</td>
<td>$2.2093 \times 10^4$</td>
<td>$3.1041 \times 10^4$</td>
</tr>
</tbody>
</table>

*Note: The minimum values of $Ra_{sp, cr}$ are printed in bold.*
We also take $H = 9 \times 10^{-12}$ W kg$^{-1}$ (by analogy with the Earth), $\bar{\rho} = 3.340$ kg m$^{-3}$, $G = 6.67 \times 10^{-11}$ N m$^2$ kg$^{-2}$, $\alpha = 3 \times 10^{-5}$ K$^{-1}$, $k = 4$ W m$^{-1}$ K$^{-1}$, $\kappa = 1$ mm$^2$ s$^{-1}$, and $Ra_{sp,cr} = 5,758$ corresponding to a rigid outer boundary. From the condition $Ra_{sp} > Ra_{sp,cr}$ for convection to occur, we find that convection would be expected within the Moon for viscosities less than $\mu = 2 \times 10^{23}$ Pas. This is about two orders of magnitude larger than the viscosity in the Earth’s mantle. Thus convection would be expected within the lunar interior, at least until volcanism removed the radiogenic heat sources.

**Question 7.4:** Is there mantle convection in the lunar interior?
7.7.2 Spherical Shells Heated Both from Within and from Below

If we consider spherical shells for which

\[ T_1 = \frac{\bar{\rho}H}{6k} \left( a^2 - c^2 \right) \]  

(7.7.45)

and \( \rho_c = \bar{\rho} \), then from (7.7.8) we find

\[ \beta_c = -\frac{\bar{\rho}H}{3k} r \]  

(7.7.46)

and (7.7.11) gives

\[ q_c(r = a) = \frac{1}{3} \bar{\rho} Ha \]  

(7.7.47)

In addition, (7.7.3) reduces to

\[ \bar{g}(r) = \frac{4}{3} \pi G \bar{\rho} r \]  

(7.7.48)

In this special case, the spherical shell is similar to the full sphere in that the acceleration of gravity and the radial temperature gradient of the conductive state are the same in the shell and the sphere. The heat flux at the outer boundary is the same for the shell and the sphere. Equation (7.7.35) which governs the onset of convection in the spherical shell reduces to (7.7.33), the equation for the sphere, and \( Ra_{sp} \) is the only dimensionless parameter, other than the radius ratio of the shell, to control the onset of convective instability in the shell.

The onset of thermal convection in this particular case of a uniformly internally heated spherical shell occurs at a value of \( Ra_{sp} \) determined by the solution of (7.7.33) subject to appropriate boundary conditions. The conditions are (7.7.42) the isothermal, impermeable conditions, applied not only at the outer boundary \( r/a = 1 \), but also at the inner boundary, \( r/a = c/a = \eta \), where we have introduced \( \eta \) to represent the radius ratio of the inner and outer boundaries. We also apply either (7.7.43) or (7.7.44) at both \( r/a = 1 \) and \( r/a = \eta \) depending on whether the boundary is rigid or shear stress free. The principle of exchange of stabilities is valid in this special spherical shell problem just as it is for the sphere (Chandrasekhar, 1961). The values of \( Ra_{sp} \) at the onset of convection in the spherical shell are denoted by \( Ra_{sp,cr} \) and are listed in Table 7.3 (after approximate results given in Chandrasekhar, 1961). The values depend on \( l \), the size of the shell \( \eta \), and the nature of the boundaries. The minimum values of \( Ra_{sp,cr} \) tend to increase and tend to occur at larger values of \( l \) with increasing \( \eta \). Thus, as the shell becomes thinner, the onset of convection tends to occur at larger spherical harmonic degrees or at shorter horizontal length scales. For a given \( \eta \), the minimum values of \( Ra_{sp,cr} \) are larger when boundaries are rigid than when they are shear stress free; for a given \( \eta \), the minimum values of \( Ra_{sp,cr} \) are also larger when \( r/a = 1 \) is rigid and \( r/a = \eta \) is shear stress free than when the reverse is the case. The larger the fraction of the boundary surface that is rigid, the larger \( Ra_{sp} \) must be to initiate convection.

The initiation of convection in the internally heated and heated from below spherical shell with shear stress free boundaries has also been studied by Schubert and Zebib (1980) for the case \( \eta = 0.5 \), a value close to that of the Earth’s mantle. Their results for \( Ra_{sp,cr} \) given in Table 7.4 are in good agreement with those of Chandrasekhar (1961). The most unstable mode has \( l = 3 \). The axisymmetric form of the \( l = 3 \) convective mode is similar to the mode of convection illustrated later in this section when we discuss the instability of a base-heated spherical shell.
Table 7.3. Approximate Values of $Ra_{sp,cr}$ (After Chandrasekhar, 1961) for the Onset of Thermal Convection in a Uniformly Internally Heated Spherical Shell That Is Also Heated at its Base;

$$\beta_c (r) = -\bar{\rho} H r / 3 k, \quad \bar{g} (r) = \frac{4}{3} \pi G \bar{\rho} r$$

<table>
<thead>
<tr>
<th>$l/\eta$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
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<td>1</td>
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<td>$6.090 \times 10^3$</td>
<td>$1.205 \times 10^4$</td>
<td>$2.996 \times 10^4$</td>
<td>$1.005 \times 10^5$</td>
<td>$5.578 \times 10^6$</td>
</tr>
<tr>
<td>2</td>
<td>$4.088 \times 10^3$</td>
<td>$5.094 \times 10^3$</td>
<td>$7.814 \times 10^3$</td>
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<td>$4.392 \times 10^4$</td>
<td>$1.972 \times 10^6$</td>
</tr>
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<td>$6.841 \times 10^3$</td>
<td>$8.566 \times 10^3$</td>
<td>$1.377 \times 10^4$</td>
<td>$3.168 \times 10^4$</td>
<td>$1.074 \times 10^6$</td>
</tr>
<tr>
<td>4</td>
<td>$1.003 \times 10^4$</td>
<td>$1.023 \times 10^4$</td>
<td>$1.135 \times 10^4$</td>
<td>$1.536 \times 10^4$</td>
<td>$2.919 \times 10^4$</td>
<td>$7.198 \times 10^5$</td>
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<tr>
<td>5</td>
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<td>$1.526 \times 10^4$</td>
<td>$1.595 \times 10^4$</td>
<td>$1.914 \times 10^4$</td>
<td>$3.089 \times 10^4$</td>
<td>$5.483 \times 10^5$</td>
</tr>
</tbody>
</table>

*a* The smallest value of $Ra_{sp,cr}$ at $\eta = 0.8$ is $3.655 \times 10^5$ for $l = 10$.

<table>
<thead>
<tr>
<th>$l/\eta$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.8</th>
</tr>
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<tbody>
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<td>$1.126 \times 10^4$</td>
<td>$2.616 \times 10^4$</td>
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<td>$4.582 \times 10^6$</td>
</tr>
<tr>
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<td>$7.477 \times 10^3$</td>
<td>$1.064 \times 10^4$</td>
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<td>$2.426 \times 10^6$</td>
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<td>$1.055 \times 10^4$</td>
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<td>$1.993 \times 10^4$</td>
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<td>$1.568 \times 10^6$</td>
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<td>$1.541 \times 10^4$</td>
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<td>$2.267 \times 10^4$</td>
<td>$4.347 \times 10^4$</td>
<td>$1.144 \times 10^6$</td>
</tr>
</tbody>
</table>

*a* The smallest value of $Ra_{sp,cr}$ at $\eta = 0.8$ is $5.843 \times 10^5$ for $l = 12$.

<table>
<thead>
<tr>
<th>$l/\eta$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.8</th>
</tr>
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<tbody>
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<td>$1.585 \times 10^4$</td>
<td>$3.189 \times 10^4$</td>
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<td>$8.401 \times 10^3$</td>
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<td>$1.739 \times 10^4$</td>
<td>$3.681 \times 10^4$</td>
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<td>$5.140 \times 10^6$</td>
</tr>
<tr>
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<td>$1.226 \times 10^4$</td>
<td>$1.617 \times 10^4$</td>
<td>$2.814 \times 10^4$</td>
<td>$7.060 \times 10^4$</td>
<td>$2.720 \times 10^6$</td>
</tr>
<tr>
<td>4</td>
<td>$1.604 \times 10^4$</td>
<td>$1.648 \times 10^4$</td>
<td>$1.893 \times 10^4$</td>
<td>$2.760 \times 10^4$</td>
<td>$5.825 \times 10^4$</td>
<td>$1.758 \times 10^6$</td>
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<tr>
<td>5</td>
<td>$2.278 \times 10^4$</td>
<td>$2.294 \times 10^4$</td>
<td>$2.446 \times 10^4$</td>
<td>$3.103 \times 10^4$</td>
<td>$5.564 \times 10^4$</td>
<td>$1.283 \times 10^6$</td>
</tr>
</tbody>
</table>

*a* The minimum value of $Ra_{sp,cr}$ at $\eta = 0.8$ is $6.520 \times 10^5$ for $l = 12$.

<table>
<thead>
<tr>
<th>$l/\eta$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.8</th>
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<td>$4.783 \times 10^5$</td>
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<td>$1.316 \times 10^4$</td>
<td>$2.402 \times 10^4$</td>
<td>$5.682 \times 10^4$</td>
<td>$1.841 \times 10^5$</td>
<td>$9.891 \times 10^6$</td>
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<td>$1.347 \times 10^4$</td>
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<td>$3.982 \times 10^4$</td>
<td>$1.128 \times 10^5$</td>
<td>$5.167 \times 10^6$</td>
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<tr>
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<td>$2.141 \times 10^4$</td>
<td>$3.571 \times 10^4$</td>
<td>$8.702 \times 10^4$</td>
<td>$3.282 \times 10^6$</td>
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<tr>
<td>5</td>
<td>$2.279 \times 10^4$</td>
<td>$2.318 \times 10^4$</td>
<td>$2.601 \times 10^4$</td>
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<td>$7.749 \times 10^4$</td>
<td>$3.245 \times 10^6$</td>
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</tbody>
</table>

*a* The minimum value of $Ra_{sp,cr}$ at $\eta = 0.6$ is $7.599 \times 10^4$ for $l = 6$.

*b* The minimum value of $Ra_{sp,cr}$ at $\eta = 0.8$ is $9.518 \times 10^5$ for $l = 13$.

*Note:* The minimum value of $Ra_{sp,cr}$ for a given $\eta$ is printed in bold.
Table 7.4. Critical Rayleigh Numbers $Ra_{sp, cr}$ for the Onset of Convection in Spherical Shells with Combined Heating and Shear Stress Free Boundaries (After Schubert and Zebib, 1980; Zebib et al., 1983)

<table>
<thead>
<tr>
<th>l/η</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
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<tbody>
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</tr>
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<td>$8.5536 \times 10^3$</td>
<td>$1.3762 \times 10^4$</td>
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</tr>
<tr>
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<td>$1.5356 \times 10^4$</td>
<td>$2.9175 \times 10^4$</td>
</tr>
<tr>
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<td>$1.5943 \times 10^4$</td>
<td>$1.9135 \times 10^4$</td>
<td>$3.0873 \times 10^4$</td>
</tr>
</tbody>
</table>

Note: Minimum values of $Ra_{sp, cr}$ are printed in bold.

The onset of thermal convection in spherical shells heated both from within and from below and with shear stress free boundaries was also considered by Zebib et al. (1983) for $\eta = 0.4$ and 0.6. Their results for $Ra_{sp, cr}$ are summarized in Table 7.4 and are in generally good agreement with the values in Table 7.3 from Chandrasekhar (1961). As $\eta$ changes from 0.4 to 0.5 to 0.6, the most unstable mode of convection changes from $l = 2$ to $l = 4$ (Table 7.4). The value of $\eta$ for the Earth’s mantle is close to 0.55, midway between the values of $\eta$ at which the preferred mode of convection onset changes from $l = 3$ to $l = 4$. The $l = 3$ axisymmetric mode of convection, illustrated later, has three cells in a meridional cross-section. The $l = 4, m = 0$ mode has four cells in a meridional plane as shown in Figure 7.7. Since there is symmetry with respect to the equatorial plane, only one hemisphere is shown. The flow shown in Figure 7.7 is actually for a supercritical value of $Ra_{sp}$ equal to $4.58 \times 10^4$, but the convection pattern is essentially similar to that at the onset of convection. The shell size in Figure 7.7 is $\eta = 0.5$, and from Table 7.4 it is seen that the $l = 4, m = 0$ mode has the lowest critical Rayleigh number among the equatorially symmetric (even) modes.

7.7.3 Spherical Shell Heated from Within

The onset of convection in a spherical shell heated only from within has been investigated by Schubert and Zebib (1980) and Zebib et al. (1983). In this circumstance the shell has an adiabatic or insulating lower boundary and the conduction temperature profile can be determined by solving (7.7.8) for the value of $T_1$ that makes $\beta_c(r = c) = 0$ with the result

$$T_c = \frac{\bar{\rho} H}{3k} \left( \frac{-r^2}{2} + \frac{c^3}{r} + \frac{a^2}{2} + \frac{c^3}{a} \right)$$

(7.7.49)

Differentiation of (7.7.49) gives

$$\beta_c(r) = \frac{\bar{\rho} H}{3k} \left( -r + \frac{c^3}{r^2} \right)$$

(7.7.50)

The stability problem for the onset of convection involves the solution of (7.7.32) with $\beta_c(r)$ given by (7.7.50) and $\bar{g}(r)$ given by (7.7.48). Schubert and Zebib (1980) and Zebib et al. (1983) considered the boundaries to be impermeable and shear stress free with the upper boundary isothermal and the lower one insulating. Thus the boundary conditions are (7.7.42) and (7.7.44) on $r/a = 1$, (7.7.44) on $r/a = \eta$, $D_1^* \theta_{lm} = 0$ on $r/a = \eta$, and $d \theta_{lm} / d(r/a) = 0$ on $r/a = \eta$. The appropriate Rayleigh number for this problem is still
7.7 Fluid Spheres and Spherical Shells

Figure 7.7. Isotherms (right) and streamlines (left) in a meridional plane for the $l = 4, m = 0$ mode at $Ra_{sp} = 4.58 \times 10^9$. Heating is both from below and from within and $\eta = 0.5$. Only one hemisphere is shown because of equatorial symmetry. There is one fast and one slow cell in each hemisphere. After Schubert and Zebib (1980).

Table 7.5. Critical Rayleigh Numbers $Ra_{sp,cr}^*$ for the Onset of Convection in a Sphere and in Spherical Shells Heated Only from Within

<table>
<thead>
<tr>
<th>$l/\eta$</th>
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<th>0.1</th>
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<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
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<td>6,616.7</td>
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<td>3,970.5</td>
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<td>3,594.7</td>
<td>4,217.7</td>
<td>5,983.4</td>
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<td>5,736.6</td>
<td>3,743.3</td>
<td>2,691.0</td>
<td>2,193.8</td>
<td>2,103.7</td>
<td>2,512.7</td>
</tr>
<tr>
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<td>26,320.6</td>
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<td>8,984.7</td>
<td>5,213.5</td>
<td>3,199.5</td>
<td>2,174.3</td>
<td>1,720.0</td>
<td>1,703.1</td>
</tr>
<tr>
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<td>14,126.9</td>
<td>7,825.5</td>
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<td>1,747.4</td>
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<tr>
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<td>8,681.0</td>
<td>4,439.1</td>
<td>2,374.0</td>
<td>1,466.5</td>
</tr>
</tbody>
</table>

Note: The minimum values of $Ra_{sp,cr}^*$ for a given $\eta$ are printed in bold.

$Ra_{sp}$ given by (7.7.34). However, to facilitate comparison with the original references we introduce the Rayleigh number $Ra_{sp}^*$ given by

$$Ra_{sp}^* = \frac{4}{3} \pi (1 - \eta)^5 Ra_{sp} \quad (7.7.51)$$

The form of the spherical shell internal heating Rayleigh number given by $Ra_{sp}^*$ is based on the value of $\bar{g}$ at $r = a$ and the thickness of the shell as the length scale.

Values of $Ra_{sp}^*$ necessary for the onset of convection, $Ra_{sp,cr}^*$, are given in Table 7.5 for modes of convection with spherical harmonic degree $l = 1$–$6$ in shells of various sizes including the sphere. The results are also presented in graphical form in Figure 7.8. The preferred form of axisymmetric convection, i.e., the convective mode with the minimum value of $Ra_{sp,cr}^*$, has one meridional cell ($l = 1$) for $0 \leq \eta \leq 0.275$, two meridional cells ($l = 2$) for $0.275 \leq \eta \leq 0.5$, and three meridional cells ($l = 3$) for $0.5 \leq \eta \leq 0.6$. Nonaxisymmetric modes of instability with $m \leq l$ are also possible; the preferred mode of convection from among the axisymmetric and nonaxisymmetric states must be determined from a nonlinear analysis. The shell sizes at which axisymmetric motions with odd and even numbers of meridional cells have the same values of $Ra_{sp,cr}^*$ offer particularly intriguing opportunities for finite-amplitude studies.
7.7.4 Spherical Shell Heated from Below

So far in our discussion of the onset of thermal convection in spherical shells, we have dealt with fluid shells that are internally heated. Here, we consider heating from below and set \( H = 0 \). We still consider the acceleration of gravity to be that appropriate to a homogeneous sphere, i.e., \( \bar{g}(r) \) is given by (7.7.48). From (7.7.8) with \( H = 0 \) we can write

\[
\beta_c(r) = -\frac{T_1}{r^2 (1/c - 1/a)} = -\frac{a^2}{r^2} \frac{\beta_c(r = a)}{r}
\]

(7.7.52)

The differential equation governing the onset of instability is (7.7.32) with \( \bar{g}(r) \) and \( \beta_c(r) \) given by (7.7.48) and (7.7.52), respectively. We can write this equation as

\[
D_l \left\{ \left( \frac{r}{a} \right)^3 D_l^* \left( \frac{\theta'_l m}{\theta_l m} \right) \right\} = -l(l + 1) R_{as, hb} \beta_l m \left( \frac{r}{a} \right)
\]

(7.7.53)

where, as before, \( D_l^* \) is the dimensionless operator and the appropriate Rayleigh number is

\[
R_{as, hb} = \frac{\alpha \bar{g} (r = a) |\beta_c(r = a)| a^4}{\kappa \nu}
\]

(7.7.54)

Solutions of (7.7.53) for isothermal, impermeable, shear stress free boundaries (conditions (7.7.42) and (7.7.44) on \( r/a = 1 \) and \( r/a = \eta \)) are given in Chandrasekhar (1961) and Zebib et al. (1980, 1983). Table 7.6 summarizes the values of \( R_{as, hb, cr} \) from Zebib et al. (1983) for the onset of thermal convection in the shell. The results are also shown in Figure 7.9. Again, to facilitate comparison with the original reference we have introduced

\[
R_{as, hb} = R_{as, hb} \frac{(1 - \eta)^4}{\eta} \quad (\eta \neq 0)
\]

(7.7.55)
7.7 Fluid Spheres and Spherical Shells

Table 7.6. Critical Rayleigh Numbers \( Ra^{*}_{sp,hb,cr} \) for the Onset of Thermal Convection in a Spherical Shell Heated Only from Below (After Zebib et al., 1983); Boundaries are Shear Stress Free and Isothermal

<table>
<thead>
<tr>
<th>( l/\eta )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3,248.0</td>
<td>1,907.2</td>
<td>1,630.2</td>
<td>1,706.1</td>
<td>2,086.5</td>
<td>2,966.6</td>
<td>5,047.2</td>
</tr>
<tr>
<td>2</td>
<td>6,288.8</td>
<td>2,423.0</td>
<td>1,450.8</td>
<td>1,133.1</td>
<td>1,095.7</td>
<td>1,300.1</td>
<td>1,941.7</td>
</tr>
<tr>
<td>3</td>
<td>13,687.3</td>
<td>4,485.8</td>
<td>2,119.9</td>
<td>1,286.0</td>
<td>978.5</td>
<td>941.6</td>
<td>1,189.6</td>
</tr>
<tr>
<td>4</td>
<td>26,303.1</td>
<td>8,287.6</td>
<td>3,489.4</td>
<td>1,780.9</td>
<td>1,109.6</td>
<td>871.9</td>
<td>917.9</td>
</tr>
<tr>
<td>5</td>
<td>45,643.5</td>
<td>14,270.5</td>
<td>5,727.7</td>
<td>2,630.8</td>
<td>1,410.2</td>
<td>928.5</td>
<td>816.1</td>
</tr>
<tr>
<td>6</td>
<td>73,544.2</td>
<td>22,962.6</td>
<td>9,054.0</td>
<td>3,915.6</td>
<td>1,885.0</td>
<td>1,072.7</td>
<td>797.2</td>
</tr>
</tbody>
</table>

Note: The minimum values of \( Ra^{*}_{sp,hb,cr} \) for a given \( \eta \) are printed in bold.

Figure 7.9. Similar to Figure 7.8 but for spherical shells heated from below. The appropriate critical Rayleigh number is denoted by \( Ra^{*}_{sp,hb,cr} \). After Zebib et al. (1983).

The Rayleigh number \( Ra^{*}_{sp,hb} \) is based on the value of \( g \) at \( r = a \), the temperature difference across the shell, and the thickness of the shell as the length scale.

When heating is only from below there is one meridional cell \((l = 1)\) at the onset of convection (with \( m = 0 \)) for shells with \( \eta \lesssim 0.275 \); there are two meridional cells \((l = 2 \text{ and } m = 0)\) for \( 0.275 \lesssim \eta \lesssim 0.46 \), three meridional cells \((l = 3 \text{ and } m = 0)\) for \( 0.46 \lesssim \eta \lesssim 0.575 \), and four meridional cells \((l = 4 \text{ and } m = 0)\) for \( 0.575 \lesssim \eta \lesssim 0.63 \), etc. There are special values of shell size indicated by the intersections of curves in Figure 7.9 for which modes with odd and even values of \( l \) have the same \( Ra^{*}_{sp,hb,cr} \). Though we have seen an example of an \( l = 1 \) axisymmetric mode in the case of an internally heated sphere in Figure 7.6, we show another example of \( l = 1 \) dominated axisymmetric convection for a sphere heated from below with \( \eta = 0.2 \) in Figure 7.10. This case is interesting because the convective cell is predominantly confined to the hemisphere in which there is upwelling at the pole. Though not at the onset of convection, the flow in Figure 7.10 is only slightly supercritical since \( Ra^{*}_{sp,hb} \) is only 2,400.
Since an \( l = 4, m = 0 \) mode of convection has already been shown in Figure 7.7 for a spherical shell with combined heating, we show here only two additional figures to illustrate the \( l = 2, m = 0 \) and \( l = 3, m = 0 \) modes of convection. The \( l = 2, m = 0 \) convection mode is shown in Figure 7.11 for heating from below, \( \eta = 0.4 \), and \( Ra_{sp, hb}^* = 4,000 \), a slightly supercritical state. Because of symmetry only one hemisphere is shown. Two distinct modes of convection are possible (as shown): one has upwelling at the pole and the other has polar downflow. Multiple convection states can occur even with the restriction to particular values of \( l \) and \( m \).

Figure 7.12 illustrates the pattern of convection associated with the \( l = 3 \) axisymmetric mode for a slightly supercritical Rayleigh number \( Ra_{sp, hb}^* = 1.6 \times 10^4 \) in a shell heated from below with \( \eta = 0.5 \) (Zebib et al., 1980). Nonaxisymmetric modes with \( l = 3 \) are also unstable but are not illustrated here. As previously noted, linear stability theory cannot resolve the selection degeneracy among the axisymmetric \( (m = 0) \) and nonaxisymmetric \( (m = 1, 2, 3) \) modes with \( l = 3 \). The \( l = 3 \) axisymmetric mode has three cells in the meridional plane (Figure 7.12). The equatorial cell rotates more rapidly than do the polar cells. Upwelling occurs at one of the poles and at about 45° latitude in the opposite hemisphere. The pattern of convection is not symmetric about the equator.
Figure 7.11. Cellular structure of a pair of equatorially symmetric motions with either rising (left) or sinking (right) at the poles. Because of symmetry only one hemisphere is shown. Heating is from below, \( \eta = 0.4 \), and \( Ra_{sp, hb} = 4,000 \). Streamlines are at the top and isotherms are at the bottom. The flows are dominated by \( l = 2 \), \( m = 0 \) modes. After Zebib et al. (1983).

### 7.8 Spherical Harmonics

Spherical geometry is of obvious relevance to studies of mantle convection and it is therefore of importance to consider the representation and solution of the equations of mass, momentum, and energy in spherical coordinates. Geophysical observables of the structure and dynamics of the mantle, e.g., the gravity field, are also conveniently described in spherical coordinates.

Any function of position on a spherical surface can be represented by a complete set of basis functions known as spherical harmonics \( Y_l^m(\theta, \phi) \) (\( \theta \) is colatitude, \( \phi \) is longitude):

\[
Y_l^m(\theta, \phi) = \left( \frac{(2l + 1) (l - m)!}{4\pi (l + m)!} \right)^{1/2} \mathcal{P}_l^m(\cos \theta) e^{im\phi} \tag{7.8.1}
\]

In (7.8.1), \( l \) and \( m \) are integers, \( l \geq 0 \) and \( -l \leq m \leq l \), and \( \mathcal{P}_l^m(\cos \theta) \) are associated Legendre functions defined in terms of the Legendre polynomials \( P_l(\cos \theta) = P_l^0(\cos \theta) \) by

\[
P_l^m(\cos \theta) = (-\sin \theta)^m \frac{d^m P_l(\cos \theta)}{d(\cos \theta)^m}, \quad 0 \leq m \leq l \tag{7.8.2}
\]

and

\[
P_l^{-m}(\cos \theta) = (-1)^m \frac{(l - m)!}{(l + m)!} P_l^m(\cos \theta) \tag{7.8.3}
\]

(Hobson, 1955; Abramowitz and Stegun, 1964). The Legendre polynomials \( P_l(\cos \theta) \) in (7.8.2) are given by

\[
P_l(\cos \theta) = \frac{1}{2^l l!} \frac{d^l}{d(\cos \theta)^l} (\cos^2 \theta - 1)^l \tag{7.8.4}
\]
From (7.8.2) it is clear that $P_l^0(\cos \theta) = P_l(\cos \theta)$. Table 7.7 summarizes some useful formulas for the Legendre polynomials and associated Legendre functions. The functions $Y_l^0 = ((2l + 1)/4\pi)^{1/2} P_l(\cos \theta)$ are functions only of latitude and accordingly are known as zonal harmonics. More generally, when $m \neq 0$ the spherical harmonics are known as tesseral harmonics except for the case $m = l$ when they are known as sectoral harmonics.

The order $m$ of the spherical harmonic gives the number of great circles passing through both poles on which $Y_l^m$ is zero. The difference between the degree $l$ of the spherical harmonic and its order $m$ gives the number of small circles parallel to the equatorial plane on which $Y_l^m$ is zero. The small circles of constant latitude and the great circles of constant longitude
### Table 7.7. Formulae Involving Legendre Polynomials and Associated Legendre Functions

#### Legendre Polynomials

\[
\frac{d P_{l+1}}{d \cos \theta} - \frac{d P_{l-1}}{d \cos \theta} - (2l + 1) P_l = 0
\]

\[(l + 1) P_{l+1} - (2l + 1) \cos \theta P_l + l P_{l-1} = 0\]

\[
\frac{d P_{l+1}}{d \cos \theta} - \cos \theta \frac{d P_l}{d \cos \theta} - (l + 1) P_l = 0
\]

\[(\cos^2 \theta - 1) \frac{d P_l}{d \cos \theta} - l \cos \theta P_l + l P_{l-1} = 0\]

\[
\frac{d^l}{d \cos \theta^l} \frac{P_l}{(2l)!} = \frac{1}{2^l l!} \left(\frac{3}{2} - \frac{(2l - 1)!!}{2^l l!}\right) l \text{ even}
\]

\[P_l(0) = 0 \quad (l \text{ odd}), \quad P_l(0) = (-1)^{l/2} \frac{\Gamma(l+1) - \Gamma(l-1)}{\sqrt{2^l}} \quad (l \text{ even})\]

\[P_l(1) = 1\]

\[P_l(-\cos \theta) = (-1)^l P_l(\cos \theta)\]

\[
\frac{d P_l}{d \cos \theta}(0) = -(l + 1) P_{l+1}(0)
\]

\[
\frac{d P_l}{d \cos \theta}(1) = \frac{l}{2} (l + 1)
\]

#### Associated Legendre Functions (Smythe, 1968⁵)

\[P_{l+1}^m = -(l + m + 1) \sin \theta P_l^m + \cos \theta P_{l+1}^{m+1}\]

\[P_{l-1}^m = -(m - l) \sin \theta P_l^m + \cos \theta P_{l-1}^{m+1}\]

\[P_{l+1}^{m+1} = (2l + 1) \cos \theta P_l^m + (m + l + 1) P_{l+1}^{m+1}\]

\[P_{l-1}^{m+1} = (m - l - 1) \sin \theta P_l^m + (2l + 1) \cos \theta P_{l-1}^m - (m + l) P_{l-1}^{m+1}\]

\[P_l^m = \frac{2m}{\sin \theta} P_{l-1}^m + (m + l - 1)(m + l) P_{l+1}^m\]

\[\sin \theta \frac{d P_l^m}{d \cos \theta} = \frac{1}{2} (m + l)(l + m + 1) P_l^{m+1} + \frac{1}{2} l P_l^{m+1}\]

\[= \sin \theta \frac{1}{\sin \theta} P_l^m - (m + l)(l + m + 1) P_l^{m+1}\]

\[= (2l + 1) \cos \theta P_l^m - (l - m + 1) P_{l+1}^m - (l - m - 1) P_{l-1}^m\]

\[= (2l + 1)^{-1} \left[(m - l - 1) P_{l+1}^m + (l + l)(m + l) P_{l-1}^m\right]\]

\[P_l^m(0) = 0 \quad (l + m \text{ odd})\]

\[P_l^m(\cos \theta) = (-1)^m (2m - 1)!! \sin^m \theta\]

\[\frac{d P_l^m}{d \cos \theta}(1) = \frac{l}{2} (l + 1)\]

\[\sin^2 \theta \frac{d P_l^m}{d \cos \theta} = \frac{1}{2} (m + l)(l + m + 1) P_l^{m+1} + \frac{1}{2} l P_l^{m+1}\]

\[= \frac{1}{\sin \theta} P_l^m - (m + l)(l + m + 1) P_l^{m+1}\]

\[= (2l + 1) \cos \theta P_l^m - (l - m + 1) P_{l+1}^m - (l - m - 1) P_{l-1}^m\]

\[= (2l + 1)^{-1} \left[(m - l - 1) P_{l+1}^m + (l + l)(m + l) P_{l-1}^m\right]\]

\[P_l^m(0) = 0 \quad (l + m \text{ odd})\]

\[P_l^m(\cos \theta) = (-1)^m (2m - 1)!! \sin^m \theta\]

<table>
<thead>
<tr>
<th>(m)</th>
<th>(P_l^m(\cos \theta))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\sin \theta)</td>
</tr>
<tr>
<td>1</td>
<td>(P_l(\cos \theta))</td>
</tr>
</tbody>
</table>

Note: The associated Legendre function defined by Smythe (1968) differs from the definition here by a minus sign when \(m\) is odd.

---

On which \(Y_l^m\) are zero are the nodal lines of \(Y_l^m\). The latitudinal nodal lines are symmetric about the equator and the longitudinal nodal lines are separated by the angle \(\pi/m\) (\(m \neq 0\)). The total number of nodal lines, both small and great circles, equals the degree \(l\) of the spherical harmonic. When \(m = 0\), the nodal lines of \(Y_l^0\) divide the spherical surface into latitudinal zones (hence the name zonal harmonics). When \(m = l\), \(Y_l^m\) is proportional to \(\sin^l \theta\) (since \(P_l(\cos \theta)\) is a polynomial in \(\cos \theta\) of maximum degree \(l\), \(d^l P_l(\cos \theta)/d(\cos \theta)^l\) is a constant, and from (7.8.2) \(Y_l^m \propto \sin^l \theta\), and there are no small circle nodal lines; there are great circle nodal lines of constant longitude that divide the spherical surface into sectors (hence the name sectoral harmonics). Figure 7.13 shows the nodal lines of several low degree and order spherical harmonics. Table 7.8 gives the functional forms of several low degree
and order spherical harmonics. Inspection of Table 7.8 illustrates that

\[ Y_{l}^{m} (\theta, \phi) = (-1)^{m} Y_{l}^{m*} (\theta, \phi) \]  

(7.8.5)

a result that can be obtained by combining (7.8.1) and (7.8.3) (the asterisk signifies the complex conjugate).

The spherical harmonic basis functions are orthonormal over the surface of a sphere

\[ \int_{0}^{2\pi} d\phi \int_{0}^{\pi} d\theta \sin \theta Y_{l}^{m} (\theta, \phi) Y_{l'}^{m'} (\theta, \phi) = \delta_{ll'} \delta_{mm'} \]  

(7.8.6)

where \( \delta_{ll'} \) is zero unless \( l = l' \) and \( \delta_{mm'} = 0 \) unless \( m = m' \). Accordingly, any function \( f(r, \theta, \phi) \) (\( r \) is radius) can be expanded according to

\[ f(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm}(r) Y_{l}^{m} (\theta, \phi) \]  

(7.8.7)

where

\[ f_{lm}(r) = \int_{0}^{2\pi} d\phi \int_{0}^{\pi} d\theta \sin \theta f(r, \theta, \phi) Y_{l}^{m*} (\theta, \phi) \]  

(7.8.8)

In addition to being a convenient set of basis functions for the surface of a sphere, spherical harmonics are the functions that arise when separable solutions of Laplace’s equation are sought in spherical coordinates. Indeed, this is the fundamental mathematical approach for the derivation of the spherical harmonics. From (6.15.24) it can be verified that if \( V \) is of the form

\[ V = c \left( r^{-l-1}, r^l \right) Y_{l}^{m} (\theta, \phi) \]  

(7.8.9)

then

\[ \nabla^2 V = c \left\{ \frac{(2l + 1)}{4\pi} \frac{(l - m)!}{(l + m)!} \right\}^{1/2} e^{i m \phi} \frac{d^2 P_{l}^{m}}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \frac{d P_{l}^{m}}{d\theta} + \left( l(l + 1) - \frac{m^2}{\sin^2 \theta} \right) P_{l}^{m} \]  

(7.8.10)
Table 7.8. Spherical Harmonics $l \leq 3$ and $-l \leq m \leq l$

<table>
<thead>
<tr>
<th>$l$</th>
<th>$P_l$</th>
<th>$P_l^m$</th>
<th>$Y_l^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$P_0^0 = 1$</td>
<td>$Y_0^0 = \frac{1}{\sqrt{4\pi}}$</td>
</tr>
<tr>
<td>1</td>
<td>$\cos \theta$</td>
<td>$P_1^{-1} = \frac{1}{2} \sin \theta$</td>
<td>$Y_1^{-1} = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{-i\phi}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_1^0 = \cos \theta$</td>
<td>$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_1^1 = -\sin \theta$</td>
<td>$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{2} \left( 3 \cos^2 \theta - 1 \right)$</td>
<td>$P_2^{-2} = \frac{1}{8} \sin^2 \theta$</td>
<td>$Y_2^{-2} = \frac{1}{8} \sqrt{\frac{30}{\pi}} \sin^2 \theta e^{-2i\phi}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_2^{-1} = \frac{1}{2} \sin \theta \cos \theta$</td>
<td>$Y_2^{-1} = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{-i\phi}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_2^0 = \frac{1}{2} \left( 3 \cos^2 \theta - 1 \right)$</td>
<td>$Y_2^0 = \frac{1}{2} \sqrt{\frac{5}{4\pi}} \left( 3 \cos^2 \theta - 1 \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_2^1 = -3 \sin \theta \cos \theta$</td>
<td>$Y_2^1 = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{i\phi}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_2^2 = 3 \sin^2 \theta$</td>
<td>$Y_2^2 = \frac{1}{8} \sqrt{\frac{30}{\pi}} \sin^2 \theta e^{2i\phi}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{2} \left( 5 \cos^3 \theta - 3 \cos \theta \right)$</td>
<td>$P_3^{-3} = \frac{\sin^3 \theta}{48}$</td>
<td>$Y_3^{-3} = \frac{1}{8} \sqrt{\frac{35}{\pi}} \sin^3 \theta e^{-3i\phi}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_3^{-2} = \frac{1}{8} \sin^2 \theta \cos \theta$</td>
<td>$Y_3^{-2} = \frac{1}{8} \sqrt{\frac{210}{\pi}} \sin^2 \theta \cos \theta e^{-2i\phi}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_3^{-1} = \frac{1}{8} \sin \theta \left( 5 \cos^2 \theta - 1 \right)$</td>
<td>$Y_3^{-1} = \frac{1}{8} \sqrt{\frac{21}{\pi}} \sin \theta \left( 5 \cos^2 \theta - 1 \right) e^{-i\phi}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_3^0 = \frac{1}{2} \left( 5 \cos^3 \theta - 3 \cos \theta \right)$</td>
<td>$Y_3^0 = \frac{1}{2} \sqrt{\frac{7}{4\pi}} \left( 5 \cos^3 \theta - 3 \cos \theta \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_3^1 = -\frac{3}{2} \left( 5 \cos^2 \theta - 1 \right) \sin \theta$</td>
<td>$Y_3^1 = -\frac{3}{8} \sqrt{\frac{7}{3\pi}} \left( 5 \cos^2 \theta - 1 \right) \sin \theta e^{i\phi}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_3^2 = 15 \cos \theta \sin^2 \theta$</td>
<td>$Y_3^2 = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \cos \theta \sin^2 \theta e^{2i\phi}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$P_3^3 = -15 \sin^3 \theta$</td>
<td>$Y_3^3 = -\frac{1}{8} \sqrt{\frac{35}{\pi}} \sin^3 \theta e^{3i\phi}$</td>
</tr>
</tbody>
</table>

The last quantity in curly brackets in (7.8.10) is identically zero and (7.8.10) is Laplace's equation

\[ \nabla^2 V = 0 \]  

(7.8.11)

i.e., the associated Legendre functions $P_l^m (\cos \theta)$ are the latitudinal functions that lead to separable solutions of Laplace's equation in spherical coordinates. The associated Legendre
functions are solutions of

\[
\frac{d^2 P_l^m}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \frac{d P_l^m}{d\theta} + \left\{ l(l+1) - \frac{m^2}{\sin^2 \theta} \right\} P_l^m = 0
\]  

(7.8.12)

The restriction of \( l \) to positive integers \( \geq 0 \) and \( m \) to \(-l \leq m \leq l\) insures that \( P_l^m \) is finite at the poles and single-valued in \( \phi \).

It is useful to write the Laplacian in terms of derivatives with respect to \( r \) and a differential operator \( L^2 \) involving only \( \theta \) and \( \phi \):

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} L^2
\]  

(7.8.13)

where

\[
L^2 \equiv -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\]  

(7.8.14)

It is straightforward to show that

\[
L^2 Y_l^m(\theta, \phi) = l(l+1) Y_l^m(\theta, \phi)
\]  

(7.8.15)

from which it follows, together with (7.8.13), that

\[
\nabla^2 f(r) Y_l^m(\theta, \phi) = Y_l^m(\theta, \phi) D_l f(r)
\]  

(7.8.16)

where

\[
D_l \equiv \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2}
\]  

(7.8.17)

We conclude this section by discussing the representation of solenoidal vector fields in terms of their toroidal and poloidal components and of the role of spherical harmonics therein. The velocity field \( \mathbf{u} \) of an incompressible flow is solenoidal as is the vorticity field \( \omega = \nabla \times \mathbf{u} \). Any solenoidal vector field can be written as the sum of a poloidal vector field \( \mathbf{S} \)

\[
\mathbf{S} = \nabla \times \left( \nabla \times \left( \frac{\Phi}{r} \right) \right) = \nabla \times \left( \nabla \left( \frac{\Phi}{r} \right) \times \mathbf{r} \right)
\]  

(7.8.18)

and a toroidal vector field \( \mathbf{T} \)

\[
\mathbf{T} = \nabla \times \left( \frac{\Psi}{r} \right) = \nabla \left( \frac{\Psi}{r} \right) \times \mathbf{r}
\]  

(7.8.19)

(Chandrasekhar, 1961). The components of \( \mathbf{S} \) and \( \mathbf{T} \) in spherical coordinates \((r, \theta, \phi)\) components) are

\[
\mathbf{S} = \left( \frac{1}{r^2} L^2 \Phi, \frac{1}{r} \frac{\partial \Phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial^2 \Phi}{\partial \phi^2} \right)
\]  

(7.8.20)

\[
\mathbf{T} = \left( 0, \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial \phi}, -\frac{1}{r} \frac{\partial \Psi}{\partial \theta} \right)
\]  

(7.8.21)
The toroidal vector field has no radial component. If $\Phi$ and $\Psi$ are expanded in terms of spherical harmonics according to (7.8.7) with radially dependent coefficients $\Phi_{lm}(r)$ and $\Psi_{lm}(r)$, then each spherical harmonic contribution to $S$ and $T$ is, respectively,

$$
S_{lm} \equiv \left( \frac{l(l+1)}{r^2} \Phi_{lm}(r) Y_{lm}^m, \frac{1}{r} \frac{d \Phi_{lm}}{dr} \frac{\partial Y_{lm}^m}{\partial \theta}, \frac{1}{r \sin \theta} \frac{d \Phi_{lm}}{dr} \frac{\partial Y_{lm}^m}{\partial \phi} \right) \quad (7.8.22)
$$

$$
T_{lm} \equiv \left( 0, \frac{\Psi_{lm}(r)}{r \sin \theta} \frac{\partial Y_{lm}^m}{\partial \phi} - \frac{\Psi_{lm}(r)}{r} \frac{\partial Y_{lm}^m}{\partial \theta} \right) \quad (7.8.23)
$$

These vector spherical harmonic fields have the following properties (Chandrasekhar, 1961). $S_{lm}$ and $T_{lm}$ are solenoidal and orthogonal to each other, i.e.,

$$
\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \ S_{lm} \cdot T_{lm} = 0 \quad (7.8.24)
$$

In addition, if $S_{lm}$ ($T_{lm}$) and $S'_{l'm'}$ ($T'_{l'm'}$) are derived from different spherical harmonics they are orthogonal in the sense of (7.8.24). If $S_{lm}$ ($T_{lm}$) and $S'_{lm}$ ($T'_{lm}$) are derived from the same spherical harmonics

$$
r^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \ S_{lm} \cdot S'_{lm} = l(l+1) \left\{ \frac{l(l+1)}{r^2} \Phi_{lm}(r) \Phi_{l'm'}(r) + \frac{d \Phi_{lm}}{dr} \frac{d \Phi_{l'm'}}{dr} \right\} \quad (7.8.25)
$$

$$
r^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \ T_{lm} \cdot T'_{lm} = l(l+1) \Psi_{lm}(r) \Psi_{l'm'}(r) \quad (7.8.26)
$$
Approximate Solutions

8.1 Introduction

The linear stability analysis presented in the last chapter gives the critical Rayleigh number for the onset of thermal convection under a variety of conditions. However, because the governing equations have been linearized, the solutions cannot predict the magnitude of finite-amplitude convective flows. In order to do this it is necessary to retain nonlinear terms in the governing equations.

Even in the simplest thermal convection problems the governing equations are sufficiently complex that analytical solutions cannot be found. There are basically two methods for obtaining nonlinear solutions. The first is to make approximations and the second is to obtain fully numerical solutions. We will address the former method in this chapter and the latter method in the subsequent two chapters.

In this chapter we will consider four approximations used to obtain a better understanding of thermal convection. We first consider an eigenmode expansion of the basic equations. This approach provides one of the methods used to obtain fully numerical solutions. However, in this chapter we consider only severe truncations of the full set of eigenmode equations. Retention of only the lowest-order nonlinear terms leads to the Lorenz (1963) equations. This set of equations is of great interest because its solution was the first demonstration of deterministic chaos. In this approximate approach we address the question:

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**Question 8.1:** Is mantle convection chaotic?

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This question leads directly to a second question:

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**Question 8.2:** Is mantle convection turbulent?

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The second approximate approach we consider is boundary layer theory. This approach reproduces the basic structure of thermal convection cells at high Rayleigh numbers. The third approach is the mean field approximation. We conclude by considering weakly nonlinear stability theory, an extension of the linear stability theory to slightly nonlinear convection.