Review of complex numbers, complex valued functions, and contour integration:

1) Find your calculus book and review things you have forgotten. In particular review integration by parts.

2) Read this chapter and do the problems marked by an *.

Reference: Advanced Calculus: Second Edition, Wilfred Kaplan, 1973.

Or refer to your own book on the topic.

Functions of a Complex Variable

In this chapter we give an introduction to the theory of analytic functions of a complex variable. The principal topics are series expansions, integrals and residues. For a more thorough treatment of these topics and a discussion of conformal mapping and its applications, the reader is referred to the author's book *Introduction to Analytic Functions* (Reading, Mass.: Addison-Wesley, 1966).

9–1 Complex functions. We assume familiarity with the complex number system (Section 0–2). Figure 9–1 (next page) reviews standard notations for the complex z-plane, where $z = x + iy$. We shall denote real and imaginary parts as follows:

$$\text{for } z = x + iy, \qquad x = \operatorname{Re} z, \qquad y = \operatorname{Im} z.$$

We shall make steady use of all these notations, as well as the analogous notations in a complex w-plane, where $w = u + iv$.

Complex-valued functions of z. If to each value of the complex number $z = x + iy$, with certain exceptions, there is assigned a value of the complex number $w = u + iv$, then w is given as a *complex-valued function* of z and we write $w = f(z)$. For example,

$$w = z^2, \qquad w = z^3 + 5z + 7, \qquad w = \frac{z+1}{z-2} \quad (z \neq 2)$$

are such functions. Important functions of this type are

polynomials: $w = a_0 z^n + \cdots + a_{n-1} z + a_n$,

rational functions: $w = \dfrac{a_0 z^n + \cdots + a_n}{b_0 z^m + \cdots + b_m}$,

exponential function: $\exp z = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$,

trigonometric functions: $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$,

hyperbolic functions: $\sinh z = \dfrac{e^z - e^{-z}}{2}, \qquad \cosh z = \dfrac{e^z + e^{-z}}{2}$,

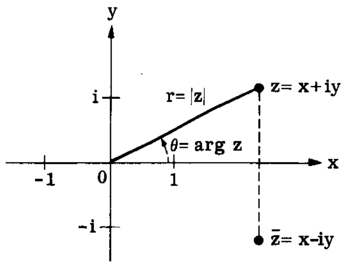

FIG. 9–1. The complex z-plane.

The definition of the exponential function is motivated by interpreting e^z for complex z as the sum of its power series $\Sigma z^n/n!$; see Section 6–19. From the definitions it follows that, for real y,

$$e^{iy} = \cos y + i \sin y \qquad \text{and} \qquad e^{-iy} = \cos y - i \sin y,$$

so that

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \qquad \cos y = \frac{e^{iy} + e^{-iy}}{2}.$$

These equations suggest the definitions given above for $\sin z$ and $\cos z$. The definitions of the hyperbolic functions are based on the usual definitions for real variables. By the reasoning described, it follows that, when z is real ($z = x + i0$), each of the five functions reduces to the familiar real function. For example, $e^{x+i0} = e^x$.

The other trigonometric and hyperbolic functions are defined in terms of the sine, cosine, sinh and cosh in the usual way.

9–2 Complex-valued functions of a real variable. It will be convenient to represent paths for line integrals in the complex plane by equations of form

$$z = F(t), \qquad a \leq t \leq b. \tag{9–1}$$

Here t is a real variable and F is a function whose values are complex, so that we are dealing with a complex-valued function of a real variable. Examples are the following functions:

$$z = e^{it}, \qquad 0 \leqq t \leqq 2\pi; \qquad z = t + it^2, \qquad 0 \leqq t \leqq 1.$$

In (9–1) we can write $z = x + iy$ and $F(t) = f(t) + ig(t)$, where f and g are real-valued. Then (9–1) is equivalent to the *pair* of equations

$$x = f(t), \qquad y = g(t), \qquad a \leqq t \leqq b. \tag{9–2}$$

We can also consider (9–1) as an alternative to the familiar vector function representation of a path.

The path $z = e^{it}$ given above is equivalent to the path $x = \cos t$, $y = \sin t$, and hence its graph is a circle.

The calculus can be developed for complex-valued functions of t in strict analogy with the development for real-valued functions. We write

$$\lim_{t \to t_0} F(t) = c = x_0 + iy_0 \tag{9–3}$$

if, given $\epsilon > 0$, we can choose $\delta > 0$ so that $|F(t) - c| < \epsilon$ when $0 < |t - t_0| < \delta$. It is assumed here that $F(t)$ is defined for t sufficiently close to t_0, but not necessarily for $t = t_0$; if $F(t)$ is defined only in an interval $t_0 < t < \beta$, the limit is interpreted as a limit *to the right* and is written

$$\lim_{t \to t_0+} F(t).$$

Limits to the left are defined similarly. Limits as $t \to \infty$ or $t \to -\infty$ are defined as for real functions. [However,

$$\lim_{t \to t_0} F(t) = \infty$$

is defined to mean

$$\lim_{t \to t_0} |F(t)| = \infty;$$

there is no concept of $+\infty$ or $-\infty$ for complex numbers; see Section 9–14.] If $F(t)$ is defined for $\alpha < t < \beta$ and t_0 lies in this interval, then $F(t)$ is said to be *continuous* at t_0 if

$$\lim_{t \to t_0} F(t) = F(t_0).$$

If $F(t)$ is also defined at $t = \alpha$ and

$$\lim_{t \to \alpha+} F(t) = F(\alpha),$$

then $F(t)$ is said to be continuous to the right at $t = \alpha$. Continuity to the left and continuity in an interval are defined as for real functions.

If $F(t) = f(t) + ig(t)$ then Eq. (9–3) is equivalent to the two equations

$$\lim_{t \to t_0} f(t) = x_0, \qquad \lim_{t \to t_0} g(t) = y_0. \tag{9–4}$$

For Eq. (9–3) signifies that $F(t)$ is as close to c as desired, for t sufficiently close to t_0; by a geometric argument we see that this is equivalent to the requirement that $f = \operatorname{Re} F$ be as close to a as desired and $g = \operatorname{Im} F$ as close to b as desired, for t sufficiently close to t_0. Similarly, continuity of $F(t)$ at t_0 is equivalent to continuity of $f(t)$ and $g(t)$ at t_0. Accordingly, such functions as $t^2 + it^3$ and e^{it} are continuous for all t. Furthermore, the rules for limits of sums, products and quotients, and the analogous continuity theorems must carry over to the complex case. For example, if $F_1(t) = f_1(t) + ig_1(t)$ and $F_2(t) = f_2(t) + ig_2(t)$ are continuous in an interval, then so also is

$$\begin{aligned} F_1(t) \cdot F_2(t) &= [f_1(t) + ig_1(t)] \cdot [f_2(t) + ig_2(t)] \\ &= f_1(t)f_2(t) - g_1(t)g_2(t) + i[f_1(t)g_2(t) + f_2(t)g_1(t)]. \end{aligned}$$

For $f_1(t)$, $g_1(t)$, $f_2(t)$, $g_2(t)$ must be continuous, so that the real and imaginary parts of $F_1(t) \cdot F_2(t)$ are continuous, and hence $F_1(t) \cdot F_2(t)$ is continuous.

For the discussion thus far, we are to some extent repeating the theory of vector functions as given in Section 0–7. However, we remark that complex multiplication and division have no analogue for vectors.

Derivatives and integrals. The derivative of $F(t)$ can be defined as for real functions:

$$F'(t_0) = \lim_{\Delta t \to 0} \frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t}. \tag{9–5}$$

By taking real and imaginary parts and applying (9–4), we conclude that

$$F'(t_0) = f'(t_0) + ig'(t_0); \tag{9–6}$$

that is, $F(t)$ has a derivative at t_0 precisely when $f(t)$, $g(t)$ have derivatives at t_0, and the derivatives are related by Eq. (9–6). Derivatives to the left or right are defined by requiring that $\Delta t < 0$ or $\Delta t > 0$, respectively in Eq. (9–5); Eq. (9–6) also applies to these derivatives.

The rules for derivative of sum, product, quotient, constant, and constant times function all carry over, and the proofs for real functions can be repeated. We also note the rules

$$\frac{d}{dt}[F(t)]^n = n[F(t)]^{n-1}F'(t) \quad (n = 1, 2, \ldots), \tag{9–7}$$

$$\frac{d}{dt} e^{(a+bi)t} = (a + bi)e^{(a+bi)t} \quad (a, b \text{ real}). \tag{9–8}$$

Furthermore, if $F'(t) \equiv 0$ for $\alpha < t < \beta$, then $F(t)$ is identically constant for $\alpha < t < \beta$. The proofs are left as exercises (Problems 9 to 11 below).

Higher derivatives are obtained by repeated differentiation:

$$F''(t) = [F'(t)]' = D^2F,\ F'''(t) = [F''(t)]' = D^3F, \ldots$$

The first derivative of $F(t)$ can be thought of as the *velocity vector* of the point (x, y) as it moves on the path $x = f(t)$, $y = g(t)$, with t as *time*. The second derivative can be interpreted as *acceleration*.

The *definite integral* of $F(t)$ over an interval $\alpha \leqq t \leqq \beta$ is defined as a limit of a sum $\sum F(t_k^*)\, \Delta_k t$ as for real functions. However, again the limit theorem permits us to take real and imaginary parts:

$$\int_\alpha^\beta F(t)\, dt = \int_\alpha^\beta f(t)\, dt + i\int_\alpha^\beta g(t)\, dt. \tag{9–9}$$

If $F(t)$ is continuous over the interval, then $f(t)$ and $g(t)$ are continuous so that the integral exists. More generally, the integral exists if $f(t)$ and $g(t)$ are *piecewise continuous* for $\alpha \leqq t \leqq \beta$. When f and g are piecewise continuous, we term $F = f + ig$ piecewise continuous.

An *indefinite integral* of $F(t)$ is defined as a function $G(t)$ whose derivative is $F(t)$. As in ordinary calculus, we find that, if $G(t)$ is one indefinite integral, then $G(t) + c$ provides all indefinite integrals:

$$\int F(t)\, dt = G(t) + c,$$

c being an arbitrary complex constant. If an indefinite integral G of F is known, then it can be used to evaluate definite integrals of F as in calculus:

$$\int_\alpha^\beta F(t)\, dt = \int_\alpha^\beta G'(t)\, dt = G(\beta) - G(\alpha). \tag{9–10}$$

The proof is left as an exercise (Problem 12 below).

From Eq. (9–9), we can verify the familiar rules for the integral of a sum, the integral of constant times function, the combination of integrals from α to β and from β to γ, and integration by parts. We also have the basic inequality

$$\left|\int_\alpha^\beta F(t)\, dt\right| \leqq \int_\alpha^\beta |F(t)|\, dt \leqq M(\beta - \alpha); \tag{9–11}$$

this is valid if $\alpha < \beta$, if $F(t)$ is, for example, piecewise continuous for $\alpha \leqq t \leqq \beta$, and if $|F(t)| \leqq M$ on this interval. The inequality is most easily obtained from the definition of the integral as limit of a sum; for we have, by repeated application of the triangle inequality [see Eq. (0–10)],

$$\left|\sum_{k=1}^n F(t_k^*)\, \Delta_k t\right| \leqq \sum_{k=1}^n |F(t_k^*)|\, \Delta_k t \leqq M(\beta - \alpha),$$

and passage to the limit gives (9–11). Definite integrals and indefinite integrals are related by the rule

$$\frac{d}{dt}\int_{\alpha}^{t} F(u)\,du = F(t) \tag{9–12}$$

(see Problem 12 below).

EXAMPLE 1. $\displaystyle\int_1^2 (t + it^2)\,dt = \left(\frac{t^2}{2} + i\frac{t^3}{3}\right)\Big|_1^2 = \frac{3}{2} + \frac{7}{3}i.$

EXAMPLE 2. $\displaystyle\int_0^1 e^{(a+bi)t}\,dt = \frac{e^{(a+bi)t}}{a+bi}\Big|_0^1 = \frac{e^{a+bi} - 1}{a+bi} \quad (a + bi \neq 0).$

EXAMPLE 3.

$$\int p(t)e^{-at}\,dt = -e^{-at}\left[\frac{p(t)}{a} + \frac{p'(t)}{a^2} + \cdots + \frac{p^{(n)}(t)}{a^{n+1}}\right] + C,$$

where $p(t)$ is a polynomial of degree n and a is a complex constant, not 0. The equation is established by integration by parts [Problem 8(f)].

Problems

* 1. (a) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Show that

$$z_1 \cdot z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

* (b) Show that $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}\cdot e^{i\theta_2}$ [see part (a)].

2. Prove the following identities:

* (a) $e^{z_1}\cdot e^{z_2} = e^{z_1+z_2}$ [see Problem 1(b)]
(b) $(e^z)^n = e^{nz}$ $(n = 0, \pm1, \pm2, \ldots)$
(c) $\sin^2 z + \cos^2 z = 1$
(d) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$
(e) $\mathrm{Re}(\sin z) = \sin x \cosh y$, $\mathrm{Im}(\sin z) = \cos x \sinh y$
(f) $\sin iz = i \sinh z$, $\cos iz = \cosh z$
(g) $\overline{e^z} = e^{\bar z}$, $\overline{\sin z} = \sin \bar z$, $\overline{\cos z} = \cos \bar z$

3. (a) Prove that $e^z \neq 0$ for all z.
(b) Prove that $\sin z$ and $\cos z$ are 0 only for appropriate real value of z.

4. Represent the following functions graphically:

(a) $w = (1 + t) + i(1 - t)$ (b) $w = t^4 + i(t^2 + 1)$
(c) $w = e^{3it}$ (d) $w = 2e^{(-1+2i)t}$
(e) $w = te^{(-1+2i)t}$ (f) $w = e^{-t} - ie^{it}$

* 5. Find the derivatives of the functions of Problem 4.

6. Graph $w = 3e^{2it}$ and indicate the first and second derivatives graphically for $t = 0$, $t = \pi/2$, $t = \pi$.

7. Integrate the functions of Problem 4 from 0 to 1.

8. Use integration by parts to evaluate each of the following:

(a) $\int (1 + it)^2 \sin t\,dt$ * (b) $\int t^n e^{-at}\,dt$ $(n = 1, 2, \ldots)$

(c) $\int t^n \sin bt\,dt = \frac{1}{2i}\int t^n(e^{bit} - e^{-bit})\,dt$ $(n = 1, 2, \ldots)$

* (d) $\int t^n \cos at\,dt = \mathrm{Re}\int t^n e^{iat}\,dt$ $(a \text{ real}, n = 1, 2, \ldots)$

(e) $\int t^n \cos at \cos bt \cos ct\,dt$ $(n = 1, 2, \ldots)$

(f) $\int p(t)e^{-at}\,dt$, where $p(t)$ is a polynomial of degree n (Example 3 in text)

9. Prove Eq. (9–7) by induction (repeated application of rule for differentiation of a product).

10. Prove (9–8) with the aid of (9–6).

11. Prove that if $F'(t) \equiv 0$, $\alpha < t < \beta$, then $F(t) \equiv$ constant for $\alpha < t < \beta$.

12. (a) Prove (9–12) by taking real and imaginary parts.
(b) Prove (9–10) either directly or as a consequence of (9–12).

Answers

5. (a) $1 - i$ (b) $4t^3 + 2it$ (c) $3ie^{3it}$ (d) $(-2 + 4i)e^{(-1+2i)t}$
(e) $e^{(-1+2i)t}[1 + t(-1 + 2i)]$ (f) $-e^{-t} + e^{it}$

6. $w' = 6ie^{2it}$, $w'' = -12e^{2it}$

7. (a) $(3 + i)/2$ (b) $(3 + 20i)/15$ (c) $(e^{3i} - 1)/3i$
(d) $2(e^{-1+2i} - 1)/(-1 + 2i)$ (e) $[1 + (2i - 2)e^{-1+2i}]/(-3 - 4i)$
(f) $2 - e^{-1} - e^i$

8. (a) $(t^2 - 2it - 3)\cos t + (2i - 2t)\sin t + c$

(b) $-e^{-at}\left(\dfrac{t^n}{a} + \dfrac{nt^{n-1}}{a^2} + \cdots + \dfrac{n!}{a^{n+1}}\right) + c$

(c) $$\cos bt\left[-\frac{t^n}{b} + \frac{n(n-1)t^{n-2}}{b^3} - \frac{n(n-1)(n-2)(n-3)t^{n-4}}{b^5} + \cdots\right] + \sin bt\left[\frac{nt^{n-1}}{b^2} - \frac{n(n-1)(n-2)t^{n-3}}{b^4} + \cdots\right] + c$$

(d) $\mathrm{Re}\left\{-e^{ait}\left[\dfrac{t^n}{-ai} + \dfrac{nt^{n-1}}{(-ai)^2} + \cdots + \dfrac{n!}{(-ai)^{n+1}}\right]\right\} + c$

(e) $-\dfrac{1}{8}\displaystyle\sum_{k=1}^{8}\left[e^{-a_kt}\left(\dfrac{t^n}{a_k} + \dfrac{nt^{n-1}}{a_k^2} + \cdots + \dfrac{n!}{a_k^{n+1}}\right)\right] + C,$

where the a_k are the 8 numbers $(\pm a \pm b \pm c)i$.

9–3 Complex-valued functions of a complex variable. Limits and continuity. We return to the general complex-valued function of a complex variable. These functions will be our principal concern for the remainder of this chapter. We write:

$$w = f(z),$$

where $z = x + iy$, $w = u + iv$, to indicate such a function. An example is the function

$$w = z^2 \quad (\text{all } z).$$

Here we can also write:

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy,$$

so that (on taking real and imaginary parts)

$$u = x^2 - y^2, \qquad v = 2xy.$$

In a similar manner, *every* complex function $w = f(z)$ is equivalent to a pair of real functions:

$$u = u(x, y) = \mathrm{Re}[f(z)], \qquad v = v(x, y) = \mathrm{Im}[f(z)],$$

of the two real variables x, y. Also from such a pair of real functions, defined on the same set, we obtain a complex function of z. For example,

$$u = x^2 + xy + y^2, \qquad v = xy^3$$

is equivalent to the complex function

$$w = f(z) = x^2 + xy + y^2 + xy^3 i,$$

for which $f(1 + 2i) = 1 + 2 + 4 + 8i = 7 + 8i$.

The functions e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ were defined in Section 9–1. For these the corresponding pairs of real functions are as follows:

$$\begin{array}{lll}
w = e^z: & u = e^x \cos y, & v = e^x \sin y, \\
w = \sin z: & u = \sin x \cosh y, & v = \cos x \sinh y, \\
w = \cos z: & u = \cos x \cosh y, & v = -\sin x \sinh y, \\
w = \sinh z: & u = \sinh x \cos y, & v = \cosh x \sin y, \\
w = \cosh z: & u = \cosh x \cos y, & v = \sinh x \sin y.
\end{array} \tag{9–13}$$

The proofs are left as exercises (Problem 1 below). In (9–13) each function is defined for all z; that is, for all (x, y).

In general, we assume $w = f(z)$ to be defined in a *domain* (open region) D in the z-plane, as suggested in Fig. 9–2. If z_0 is a point of D, we can then find a circular *neighborhood* $|z - z_0| < k$ about z_0 in D. If $f(z)$ is defined in such a neighborhood, except perhaps at z_0, then we write

$$\lim_{z \to z_0} f(z) = w_0 \tag{9–14}$$

if, for every $\epsilon > 0$, we can choose $\delta > 0$, so that

$$|f(z) - w_0| < \epsilon \quad \text{for } 0 < |z - z_0| < \delta. \tag{9–15}$$

If $f(z_0)$ is defined and equals w_0, and (9–14) holds, then we call $f(z)$ *continuous at* z_0.

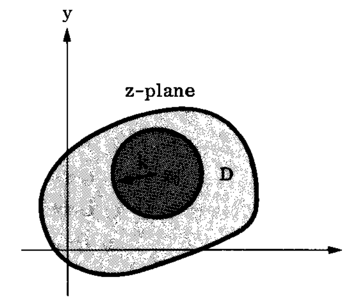

FIG. 9–2. Domain and neighborhood.

THEOREM 1. *The function $w = f(z)$ is continuous at $z_0 = x_0 + iy_0$ if and only if $u(x, y) = \mathrm{Re}\,[f(z)]$ and $v(x, y) = \mathrm{Im}\,[f(z)]$ are continuous at (x_0, y_0).*

Thus $w = z^2 = x^2 - y^2 + 2ixy$ is continuous for all z, since $u = x^2 - y^2$ and $v = 2xy$ are continuous for all (x, y). The proof of Theorem 1 is left as an exercise (Problem 5 below).

THEOREM 2. *The sum, product, and quotient of continuous functions of z are continuous, except for division by zero; a continuous function of a continuous function is continuous. Similarly, if the limits exist,*

$$\lim_{z \to z_0} [f(z) + g(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z), \quad \ldots \tag{9–16}$$

These properties are proved as for real variables. (It is assumed in Theorem 2 that the functions are defined in appropriate domains.)

It follows from Theorem 2 that polynomials in z are continuous for all z, and each rational function is continuous except where the denominator is zero. From Theorem 1 it follows that

$$e^z = e^x \cos y + ie^x \sin y$$

is continuous for all z. Hence, by Theorem 2, so also are the functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

We write

$$\lim_{z \to z_0} f(z) = \infty \quad \text{if } \lim_{z \to z_0} |f(z)| = +\infty;$$

that is, if for each real number K there is a positive δ such that $|f(z)| > K$

for $0 < |z - z_0| < \delta$. Similarly, if $f(z)$ is defined for $|z| > R$, for some R, then $\lim_{z\to\infty} f(z) = c$ if for each $\epsilon > 0$ we can choose a number R_0 such that $|f(z) - c| < \epsilon$ for $|z| > R_0$. All these definitions emphasize that there is but *one* complex number ∞ and that "approaching ∞" is equivalent to receding from the origin.

9–4 Derivatives and differentials. Let $w = f(z)$ be given in D and let z_0 be a point of D. Then w is said to have a derivative $f'(z_0)$ if

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f'(z_0).$$

In appearance this definition is the same as that for functions of a real variable, and it will be seen that the derivative does have the usual properties. However, it will also be shown that if $w = f(z)$ has a continuous derivative in a domain D, then $f(z)$ has a number of additional properties; in particular, the second derivative $f''(z)$, third derivative $f'''(z), \ldots,$ must also exist in D.

The reason for the remarkable consequences of possession of a derivative lies in the fact that the increment Δz is allowed to approach zero in any manner. If we restricted Δz so that $z_0 + \Delta z$ approached z_0 along a particular line, then we would obtain a "directional derivative." But here the limit obtained is required to be the *same for all directions*, so that the "directional derivative" has the same value in all directions. Moreover, $z_0 + \Delta z$ may approach z_0 in a quite arbitrary manner, for example along a spiral path. The limit of the ratio $\Delta w/\Delta z$ must be the same for all manners of approach.

We say that $f(z)$ has a *differential* $dw = c\,\Delta z$ at z_0 if $f(z_0 + \Delta z) - f(z_0) = c\,\Delta z + \epsilon\,\Delta z$, where ϵ depends on Δz and is continuous at $\Delta z = 0$, with value zero when $\Delta z = 0$.

THEOREM 3. *If $w = f(z)$ has a differential $dw = c\,\Delta z$ at z_0, then w has a derivative $f'(z_0) = c$. Conversely, if w has a derivative at z_0, then w has a differential at z_0: $dw = f'(z_0)\,\Delta z$.*

This is proved just as for real functions. We also write $\Delta z = dz$, as for real variables, so that

$$dw = f'(z)\,dz, \qquad \frac{dw}{dz} = f'(z). \tag{9–17}$$

From Theorem 3 we see that existence of the derivative $f'(z_0)$ implies continuity of f at z_0, for

$$f(z_0 + \Delta z) - f(z_0) = c\,\Delta z + \epsilon\,\Delta z \to 0$$

as $\Delta z \to 0$.

THEOREM 4. *If w_1 and w_2 are functions of z which have differentials in D, then*

$$d(w_1 + w_2) = dw_1 + dw_2,$$
$$d(w_1 w_2) = w_1\,dw_2 + w_2\,dw_1, \tag{9–18}$$
$$d\frac{w_1}{w_2} = \frac{w_2\,dw_1 - w_1\,dw_2}{w_2^2} \quad (w_2 \neq 0).$$

If w_2 is a differentiable function of w_1, and w_1 is a differentiable function of z, then wherever $w_2[w_1(z)]$ is defined

$$\frac{dw_2}{dz} = \frac{dw_2}{dw_1}\cdot\frac{dw_1}{dz}. \tag{9–19}$$

These rules are proved as in elementary calculus. We can now prove as usual the basic rule:

$$\frac{d}{dz} z^n = nz^{n-1} \quad (n = 1, 2, \ldots). \tag{9–20}$$

Furthermore, the derivative of a constant is zero.

Problems

1. For each of the following write the given function as two real functions of x and y and determine where the given function is continuous:

(a) $w = (1 + i)z^2$ (b) $w = \dfrac{z}{z + i}$

(c) $w = \tan z = \dfrac{\sin z}{\cos z}$ (d) $w = \dfrac{e^{-z}}{z + 1}$

*(e) $w = e^z$ (f) $w = \sin z$

*(g) $w = \cos z$ (h) $w = \sinh z$

*(i) $w = \cosh z$ (j) $w = e^z \cos z$

2. Evaluate each of the following limits:

(a) $\lim\limits_{z\to\pi i} \dfrac{\sin z + z}{e^z + 2}$ (b) $\lim\limits_{z\to 0} \dfrac{z^2 - z}{2z}$

(c) $\lim\limits_{z\to 0} \dfrac{\cos z}{z}$ (d) $\lim\limits_{z\to\infty} \dfrac{z}{z^2 + 1}$

3. Differentiate each of the following complex functions:

*(a) $w = z^3 + 5z + 1$ (b) $w = \dfrac{1}{z - 1}$

(c) $w = [1 + (z^2 + 1)^3]^7$ (d) $w = \dfrac{z^2}{(z + 1)^3}$

4. Prove the rule (9–20).

5. Prove Theorem 1.

Answers

1. (a) $u = x^2 - y^2 - 2xy$, $v = x^2 - y^2 + 2xy$, all z
(b) $u = (x^2 + y^2 + y)[x^2 + (y+1)^2]^{-1}$
$v = -x[x^2 + (y+1)^2]^{-1}$ $(z \neq -i)$
(c) $u = \tan x \operatorname{sech}^2 y\,[1 + \tan^2 x \tanh^2 y]^{-1}$
$v = \tanh y \sec^2 x\,[1 + \tan^2 x \tanh^2 y]^{-1}$
$z \neq (\pi/2) + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$
(d) $u = e^{-x}[(1+x)\cos y - y \sin y][(1+x)^2 + y^2]^{-1}$
$v = -e^{-x}[(1+x)\sin y + y\cos y][(1+x)^2 + y^2]^{-1}$ $(z \neq -1)$
(e) . . . (i) See (9–13), continuous for all z
(j) $u = e^x(\cos x \cos y \cosh y + \sin x \sin y \sinh y)$
$v = e^x(\cos x \sin y \cosh y - \sin x \cos y \sinh y)$, all z

2. (a) $i(\pi + \sinh \pi)$ (b) $-\frac{1}{2}$ (c) ∞ (d) 0

3. (a) $3z^2 + 5$ (b) $-(z-1)^{-2}$ (c) $42[1 + (z^2+1)^3]^6(z^2+1)^2 z$
(d) $(z+1)^{-4}(2z - z^2)$

9–5 Integrals. The complex integral $\int f(z)\,dz$ is defined as a line integral, and its properties are closely related to those of the integral $\int P\,dx + Q\,dy$ (see Chapter 5).

Let C be a path from A to B in the complex plane: $x = x(t)$, $y = y(t)$, $a \leqq t \leqq b$. We assume C to have a direction, usually that of increasing t. We subdivide the interval $a \leqq t \leqq b$ into n parts by $t_0 = a, t_1, \ldots, t_n = b$. We let $z_j = x(t_j) + iy(t_j)$ and $\Delta_j z = z_j - z_{j-1}$, $\Delta_j t = t_j - t_{j-1}$. We choose an arbitrary value t_j^* in the interval $t_{j-1} \leqq t \leqq t_j$ and set $z_j^* = x(t_j^*) + iy(t_j^*)$. These quantities are all shown in Fig. 9–3. We then write

$$\int\limits_C f(z)\,dz = \int\limits_{C\;A}^{B} f(z)\,dz = \lim_{\substack{n\to\infty\\ \max \Delta_j t \to 0}} \sum_{j=1}^{n} f(z_j^*)\,\Delta_j z. \tag{9–21}$$

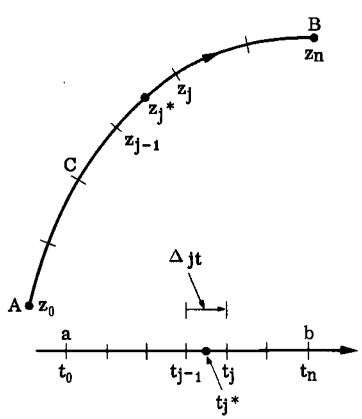

FIG. 9–3. Complex line integral.

If we take real and imaginary parts in (9–21), we find

$$\int\limits_C f(z)\,dz = \lim \sum (u + iv)\,(\Delta x + i\,\Delta y)$$
$$= \lim \{\sum(u\,\Delta x - v\,\Delta y) + i\sum(v\,\Delta x + u\,\Delta y)\};$$

that is,

$$\int\limits_C f(z)\,dz = \int\limits_C (u + iv)\,(dx + i\,dy)$$
$$= \int\limits_C (u\,dx - v\,dy) + i\int\limits_C (v\,dx + u\,dy). \tag{9–22}$$

The complex line integral is thus simply a combination of two real line integrals. Hence we can apply all the theory of real line integrals. In the following, each path is assumed to be *piecewise smooth;* that is, $x(t)$ and $y(t)$ are to be continuous with piecewise continuous derivatives.

THEOREM 5. *If $f(z)$ is continuous in domain D, then the integral* (9–21) *exists and*

$$\int\limits_C f(z)\,dz = \int_a^b \left(u\frac{dx}{dt} - v\frac{dy}{dt}\right)dt + i\int_a^b \left(v\frac{dx}{dt} + u\frac{dy}{dt}\right)dt. \tag{9–23}$$

We now write our path as $z = z(t)$ as in Section 9–2. If we introduce the derivative

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$$

of z with respect to the real variable t, and also use the theory of integrals of such functions (Section 9–2), we can write (9–23) more concisely:

$$\int\limits_C f(z)\,dz = \int_a^b f[z(t)]\frac{dz}{dt}\,dt. \tag{9–24}$$

EXAMPLE 1. Let C be the path $x = 2t$, $y = 3t$, $1 \leqq t \leqq 2$. Let $f(z) = z^2$. Then

$$\int\limits_C z^2\,dz = \int_1^2 (2t + 3it)^2(2 + 3i)\,dt = (2+3i)^3\int_1^2 t^2\,dt$$
$$= -107\tfrac{1}{3} + 21i.$$

EXAMPLE 2. Let C be the circular path $x = \cos t$, $y = \sin t$, $0 \leqq t \leqq 2\pi$. This can be written more concisely thus: $z = e^{it}$, $0 \leqq t \leqq 2\pi$. Since $dz/dt = ie^{it}$,

$$\int\limits_C \frac{1}{z}\,dz = \int_0^{2\pi} e^{-it}ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Further properties of complex integrals follow from those of real integrals:

THEOREM 6. *Let $f(z)$ and $g(z)$ be continuous in a domain D. Let C be a piecewise smooth path in D. Then*

$$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz,$$

$$\int_C kf(z)\,dz = k\int_C f(z)\,dz \quad (k = \text{const}),$$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz,$$

where C is composed of a path C_1 from z_0 to z_1 and a path C_2 from z_1 to z_2, and

$$\int_C f(z)\,dz = -\int_{C'} f(z)\,dz,$$

where C' is obtained from C by reversing direction on C.

Upper estimates for the absolute value of a complex integral are obtained by the following theorem.

THEOREM 7. *Let $f(z)$ be continuous on C, let $|f(z)| \leqq M$ on C, and let*

$$L = \int_C ds = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$$

be the length of C. Then

$$\left|\int_C f(z)\,dz\right| \leqq \int_C |f(z)|\,ds \leqq M \cdot L. \tag{9–25}$$

Proof. The line integral $\int |f(z)|\,ds$ is defined as a limit:

$$\int_C |f(z)|\,ds = \lim \sum |f(z_j^*)|\,\Delta_j s,$$

where $\Delta_j s$ is the length of the jth arc of C. Now

$$|f(z_j^*)\,\Delta_j z| = |f(z_j^*)| \cdot |\Delta_j z| \leqq |f(z_j^*)| \cdot \Delta_j s,$$

for $|\Delta_j z|$ represents the *chord* of the arc $\Delta_j s$. Hence

$$|\sum f(z_j^*)\,\Delta_j z| \leqq \sum |f(z_j^*)\,\Delta_j z| \leqq \sum |f(z_j^*)|\,\Delta_j s$$

by repeated application of the triangle inequality (0–10). Passing to the limit, we conclude that

$$\left|\int_C f(z)\,dz\right| \leqq \int_C |f(z)|\,ds. \tag{9–26}$$

Also, if $|f| \leqq M = \text{const}$,

$$\sum |f(z_j^*)|\,\Delta_j s \leqq \sum M\,\Delta_j s = M \cdot L.$$

Hence

$$\int_C |f(z)|\,ds \leqq M \cdot L. \tag{9–27}$$

Inequalities (9–25) follow from (9–26) and (9–27).

Problems

1. Evaluate the following integrals:

(a) $\int_0^{1+i} (x^2 - iy^2)\,dz$ on the straight line from 0 to $1 + i$.

*(b) $\int_0^{\pi} z\,dz$ on the curve $y = \sin x$. (c) $\int_1^{1+i} \frac{dz}{z}$ on the line $x = 1$.

2. Write each of the following integrals in the form $\int u\,dx - v\,dy + i\int v\,dx + u\,dy$; then show that each of the two real integrals is independent of path in the xy-plane.

(a) $\int (z + 1)\,dz$ *(b) $\int e^z\,dz$
(c) $\int z^4\,dz$ (d) $\int \sin z\,dz$

*3. (a) Evaluate

$$\oint \frac{1}{z}\,dz$$

on the circle $|z| = R$.

*(b) Show that

$$\oint \frac{1}{z}\,dz = 0$$

on every simple closed path not meeting or enclosing the origin.

(c) Show that

$$\oint \frac{1}{z^2}\,dz = 0$$

on every simple closed path not passing through the origin.

Answers

1. (a) $\frac{2}{3}$ (b) $\pi^2/2$ (c) $\frac{1}{2}\log 2 + i(\pi/4)$
3. (a) $2\pi i$

9–6 Analytic functions. Cauchy-Riemann equations. A function $w = f(z)$, defined in a domain D, is said to be an *analytic function* in D if w has a continuous derivative in D. Almost the entire theory of functions of a complex variable is confined to the study of such functions. Furthermore, almost all functions used in the applications of mathematics to physical problems are analytic functions or are derived from such.

It will be seen that possession of a continuous derivative implies possession of a continuous second derivative, third derivative, . . . , and, in fact, convergence of the Taylor series

$$f(z_0) + f'(z_0)\,\frac{(z - z_0)}{1!} + f''(z_0)\,\frac{(z - z_0)^2}{2!} + \cdots$$

in a neighborhood of each z_0 of D. Thus one could define an analytic function as one so representable by Taylor series, and this definition is often used. The two definitions are equivalent, for convergence of the Taylor series in a neighborhood of each z_0 implies continuity of the derivatives of all orders.

While it is possible to construct continuous functions of z which are not analytic (examples will be given below), it is impossible to construct a function $f(z)$ possessing a derivative, but not a continuous one, in D. In other words, if $f(z)$ has a derivative in D, the derivative is necessarily continuous, so that $f(z)$ is analytic. Therefore we could define an analytic function as one merely possessing a derivative in domain D, and this definition is also often used. For a proof that existence of the derivative implies its continuity, refer to Vol. I of the book by Knopp listed at the end of the chapter.

THEOREM 8. *If $w = u + iv = f(z)$ is analytic in D, then u and v have continuous first partial derivatives in D and satisfy the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{9–28}$$

in D. Furthermore,

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{9–29}$$

Conversely, if $u(x, y)$ and $v(x, y)$ have continuous first partial derivatives in D and satisfy the Cauchy-Riemann equations (9–28), *then $w = u + iv = f(z)$ is analytic in D.*

Proof. Let z_0 be a fixed point of D and let

$$\Delta w = \Delta u + i\,\Delta v = f(z_0 + \Delta z) - f(z_0), \qquad \Delta z = \Delta x + i\,\Delta y,$$

as in Fig. 9–4. We consider several equivalent formulations of the condition that $f'(z_0)$ exists. Throughout, ϵ, ϵ_1, ϵ_2, ϵ_3, ϵ_4 denote functions of $\Delta z = \Delta x + i\,\Delta y$, continuous and equal to zero at $\Delta z = 0$. By Theorem 3, existence of $f'(z_0)$ is equivalent to the statement

$$\Delta w = c \cdot \Delta z + \epsilon \cdot \Delta z, \qquad c = f'(z_0), \qquad c = a + ib; \tag{9–30}$$

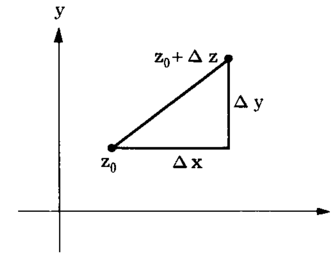

FIG. 9–4. Complex derivative.

this is equivalent to

$$\Delta w = c\,\Delta z + \epsilon\,\Delta x + i\epsilon\,\Delta y \tag{9–30′}$$

and also to

$$\Delta w = c\,\Delta z + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y + i(\epsilon_3\,\Delta x + \epsilon_4\,\Delta y), \tag{9–30″}$$

where ϵ_1, ϵ_2, ϵ_3, ϵ_4 are real. For if (9–30′) holds, then (9–30″) holds with $\epsilon_1 = \mathrm{Re}\,(\epsilon)$, $\epsilon_2 = -\mathrm{Im}\,(\epsilon)$, $\epsilon_3 = \mathrm{Im}\,(\epsilon)$, $\epsilon_4 = \mathrm{Re}\,(\epsilon)$. Conversely, if (9–30″) holds, then (9–30′) holds with $\epsilon = 0$ for $\Delta z = 0$ and

$$\epsilon = (\epsilon_1 + i\epsilon_3)\,\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\epsilon_4)\,\frac{\Delta y}{\Delta z} \quad (\Delta z \neq 0). \tag{9–31}$$

As Fig. 9–4 shows,

$$\left|\frac{\Delta x}{\Delta z}\right| \leqq 1, \qquad \left|\frac{\Delta y}{\Delta z}\right| \leqq 1,$$

so that we deduce from (9–31) that $\epsilon \to 0$ as $\Delta z \to 0$. Thus (9–30), (9–30′) and (9–30″) are all equivalent to existence of $f'(z_0) = c = a + ib$. By taking real and imaginary parts in (9–30″), we obtain one more equivalent condition:

$$\begin{aligned} \Delta u &= a\,\Delta x - b\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y,\\ \Delta v &= b\,\Delta x + a\,\Delta y + \epsilon_3\,\Delta x + \epsilon_4\,\Delta y; \end{aligned} \tag{9–30‴}$$

these equations state that u, v have differentials $du = a\,dx - b\,dy$, $dv = b\,dx + a\,dy$ at (x_0, y_0), and hence at this point

$$\frac{\partial u}{\partial x} = a = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -b = -\frac{\partial v}{\partial x}.$$

Thus differentiability of $f'(z)$ at any z is equivalent to differentiability of u, v along with validity of the Cauchy-Riemann equations. Furthermore, $f'(z)$ and $\partial u/\partial x, \ldots$ are related by (9–29). By Theorem 1, these equations show that continuity of $f'(z)$ in D is equivalent to continuity of $\partial u/\partial x, \ldots$ Thus the theorem is proved.

The theorem provides a perfect test for analyticity: if $f(z)$ is analytic, then the Cauchy-Riemann equations hold; if the equations hold (and the derivatives concerned are continuous), then $f(z)$ is analytic.

EXAMPLE 1. $w = z^2 = x^2 - y^2 + i \cdot 2xy$. Here $u = x^2 - y^2, v = 2xy$. Thus

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

and w is analytic for all z.

EXAMPLE 2. $w = \dfrac{x}{x^2 + y^2} - \dfrac{iy}{x^2 + y^2}$. Here

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}.$$

Hence w is analytic except for $x^2 + y^2 = 0$, that is, for $z = 0$.

EXAMPLE 3. $w = x - iy = \bar{z}$. Here $u = x$, $v = -y$ and

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1, \quad \frac{\partial u}{\partial y} = 0 = \frac{\partial v}{\partial x}.$$

Thus w is not analytic in any domain.

EXAMPLE 4. $w = x^2y^2 + 2x^2y^2i$. Here

$$\frac{\partial u}{\partial x} = 2xy^2, \quad \frac{\partial v}{\partial y} = 4x^2y, \quad \frac{\partial u}{\partial y} = 2x^2y, \quad \frac{\partial v}{\partial x} = 4xy^2.$$

The Cauchy-Riemann equations give $2xy^2 = 4x^2y$, $2x^2y = -4xy^2$. These equations are satisfied only along the lines $x = 0$, $y = 0$. There is *no domain* in which the Cauchy-Riemann equations hold, hence no domain in which $f(z)$ is analytic. One does not consider functions analytic only at certain points unless these points form a domain.

The terms "analytic at a point" or "analytic along a curve" are used, apparently in contradiction to the remark just made. However, we say that $f(z)$ is *analytic at the point* z_0 only if there is a domain containing z_0 within which $f(z)$ is analytic. Similarly, $f(z)$ is *analytic along a curve* C only if $f(z)$ is analytic in a domain containing C.

THEOREM 9. *The sum, product, and quotient of analytic functions is analytic (provided in the last case the denominator is not equal to zero at any point of the domain under consideration). All polynomials are analytic for all z. Every rational function is analytic in each domain containing no root of the denominator. An analytic function of an analytic function is analytic.*

This follows from Theorem 4.

We readily verify (Problem 1 below) that the Cauchy-Riemann equations are satisfied for $u = \text{Re}(e^z)$, $v = \text{Im}(e^z)$. Hence e^z is analytic for all z. It then follows from Theorem 9 that $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ are analytic for all z, while $\tan z$, $\sec z$, and $\csc z$ are analytic except for certain points (Problem 6 below). Furthermore, the usual formulas for derivatives hold:

$$\frac{d}{dz} e^z = e^z, \quad \frac{d}{dz} \sin z = \cos z, \quad \ldots \tag{9-32}$$

(Problem 3).

Two basic theorems of more advanced theory are useful at this point. Proofs are given in Chapter IV of the book by Goursat listed at the end of the chapter.

THEOREM 10. *Given a function $f(x)$ of the real variable x, $a \leqq x \leqq b$, there is at most one analytic function $f(z)$ which reduces to $f(x)$ when z is real.*

THEOREM 11. *If $f(z)$, $g(z)$, ... are functions which are all analytic in a domain D which includes part of the real axis, and $f(z)$, $g(z)$, ... satisfy an algebraic identity when z is real, then these functions satisfy the same identity for all z in D.*

Theorem 10 implies that our definitions of e^z, $\sin z$, ... are the only ones which yield analytic functions and agree with the definitions for real variables.

Because of Theorem 11, we can be sure that all familiar identities of trigonometry, namely,

$$\sin^2 z + \cos^2 z = 1, \quad \sin\left(\frac{\pi}{2} - z\right) = \cos z, \quad \ldots \tag{9-33}$$

continue to hold for complex z. A general algebraic identity is formed by replacing the variables $w_1, \ldots, w_n$ in an algebraic equation by functions $f_1(z), \ldots, f_n(z)$. Thus, in the two examples given, one has

$$w_1^2 + w_2^2 - 1 = 0 \quad (w_1 = \sin z, w_2 = \cos z),$$

$$w_1 - w_2 = 0 \quad \left[w_1 = \sin\left(\frac{\pi}{2} - z\right), w_2 = \cos z\right].$$

To prove identities such as

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}, \tag{9-34}$$

it may be necessary to apply Theorem 11 several times. (See Problems 4 and 5 below.)

It should be remarked that while e^z is written as a power of e, it is best not to think of it as such. Thus $e^{1/2}$ has only one value, not two, as would a usual complex root. To avoid confusion with the general power function, to be defined below, we often write $e^z = \exp z$ and refer to e^z as the *exponential function of z*.

To obtain the real and imaginary parts of $\sin z$, we use the identity

$$\sin (z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$

which holds, by the reasoning described above, for all complex z_1 and z_2. Hence $\sin (x + iy) = \sin x \cos iy + \cos x \sin iy$. Now from the definitions (Section 9–1),

$$\sinh y = -i \sin iy, \qquad \cosh y = \cos iy. \tag{9–35}$$

Hence

$$\sin z = \sin x \cosh y + i \cos x \sinh y. \tag{9–36}$$

Similarly, we prove, as in (9–13) above,

$$\begin{aligned} \cos z &= \cos x \cosh y - i \sin x \sinh y, \\ \sinh z &= \sinh x \cos y + i \cosh x \sin y, \\ \cosh z &= \cosh x \cos y + i \sinh x \sin y. \end{aligned} \tag{9–37}$$

Conformal mapping. A complex function $w = f(z)$ can be considered as a *mapping* from the xy-plane to the uv-plane as in Section 2–7. In the case of an analytic function $f(z)$, this mapping has a special property: it is a *conformal mapping*. By this we mean that two curves in the xy-plane, meeting at (x_0, y_0) at angle α, correspond to two curves meeting at the corresponding point (u_0, v_0) at the *same angle* α (in value and in sense—positive or negative). This means that a small triangle in the xy-plane corresponds to a *similar* small (curvilinear) triangle in the uv-plane. (The properties described fail at the exceptional points where $f'(z) = 0$.) Furthermore, every conformal mapping from the xy-plane to the uv-plane is given by an analytic function. For a discussion of conformal mapping and its applications, see Chapter 7 of the book by Kaplan listed at the end of the chapter.

Problems

1. Verify that the following are analytic functions of z:

(a) $2x^3 - 3x^2y - 6xy^2 + y^3 + i(x^3 + 6x^2y - 3xy^2 - 2y^3)$
(b) $w = e^z = e^x \cos y + ie^x \sin y$
(c) $w = \sin z = \sin x \cosh y + i \cos x \sinh y$

2. Test each of the following for analyticity:

(a) $x^3 + y^3 + i(3x^2y + 3xy^2)$ (b) $\sin x \cos y + i \cos x \sin y$
(c) $3x + 5y + i(3y - 5x)$

3. Prove the following properties directly from the definitions of the functions:

(a) $\dfrac{d}{dz} e^z = e^z$ (b) $\dfrac{d}{dz} \sin z = \cos z$, $\dfrac{d}{dz} \cos z = -\sin z$
(c) $\sin (z + \pi) = -\sin z$ (d) $\sin (-z) = -\sin z$, $\cos (-z) = \cos z$

4. Prove the identity $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ by application of Theorem 11. [*Hint:* Let $z_2 = b$, a fixed real number, and $z_1 = z$, a variable complex number. Then $e^{z+b} = e^z \cdot e^b$ is an identity connecting analytic functions which is known to be true for z real. Hence it is true for all complex z. Now proceed similarly with the identity $e^{z_1+z} = e^{z_1} \cdot e^z$.]

5. Prove the following identities by application of Theorem 11 (see Problem 4):

(a) $\cos (z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$
(b) $e^{iz} = \cos z + i \sin z$
(c) $(e^z)^n = e^{nz}$ $(n = 0, 1, 2, \ldots)$

6. Determine where the following functions are analytic (see Problem 3 following Section 9–2):

(a) $\tan z = \dfrac{\sin z}{\cos z}$ (b) $\cot z = \dfrac{\cos z}{\sin z}$
(c) $\tanh z = \dfrac{\sinh z}{\cosh z}$ (d) $\dfrac{\sin z}{z}$
(e) $\dfrac{e^z}{z \cos z}$ (f) $\dfrac{e^z}{\sin z + \cos z}$

Answers

2. (a) Analytic nowhere, (b) analytic nowhere, (c) analytic for all z.
6. The functions are analytic except at the following points: (a) $\frac{1}{2}\pi + n\pi$; (b) $n\pi$; (c) $\frac{1}{2}\pi i + n\pi i$; (d) 0; (e) 0, $\frac{1}{2}\pi + n\pi$; (f) $-\frac{1}{4}\pi + n\pi$, where $n = 0, \pm1, \pm2, \ldots$

9–7 The functions log z, a^z, z^a, $\sin^{-1} z$, $\cos^{-1} z$. The function $w = \log z$ is defined as the inverse of the exponential function $z = e^w$. We write $z = re^{i\theta}$, in terms of polar coordinates r, θ, and $w = u + iv$, so that

$$re^{i\theta} = e^{u+iv} = e^u e^{iv},$$

$$e^u = r, \qquad v = \theta + 2k\pi \quad (k = 0, \pm1, \ldots).$$

Accordingly,

$$w = \log z = \log r + i(\theta + 2k\pi) = \log |z| + i \arg z, \tag{9–38}$$

where $\log r$ is the real logarithm of r Thus $\log z$ is a multiple-valued function of z, with infinitely many values except for $z = 0$. We can select one value of θ for each z and obtain a single-valued function, $\log z = \log r + i\theta$; however, θ cannot be chosen to depend continuously on z for all $z \neq 0$, since θ will increase by 2π each time one encircles the origin in the positive direction.

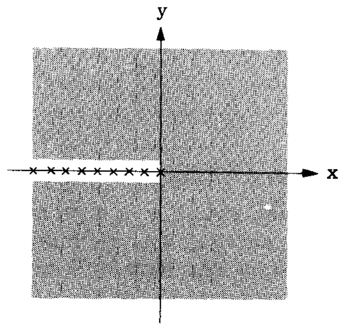

FIG. 9–5. Domain for log z.

If we concentrate on an appropriate portion of the z-plane, we can choose θ to vary continuously within the domain. For example, the inequalities

$$-\pi < \theta < \pi, \qquad r > 0$$

together describe a domain (Fig. 9–5) and also tell how to assign the values of θ within the domain. With θ so restricted, $\log r + i\theta$ then defines a *branch* of $\log z$ in the domain chosen; this particular branch is called the *principal value* of $\log z$ and is denoted by $\operatorname{Log} z$. The points on the negative real axis are excluded from the domain, but we usually assign the values $\operatorname{Log} z = \log |x| + i\pi$ on this line. Within the domain of Fig. 9–5, $\operatorname{Log} z$ is *an analytic function of* z (Problem 4 below). Other branches of $\log z$ are obtained by varying the choice of θ or of the domain. For example, in the domain of Fig. 9–5, we might choose θ so that $\pi < \theta < 3\pi$, or so that $-3\pi < \theta < -\pi$. The inequalities $0 < \theta < 2\pi$, $\pi/2 < \theta < 5\pi/2, \ldots$ also suggest other domains and choices of θ. We can verify that so long as θ varies continuously in the domain, $\log z = \log r + i\theta$ is analytic there. The most general domain possible here is an arbitrary simply-connected domain not containing the origin.

As a result of this discussion, it appears that $\log z$ is formed of many branches, each analytic in some domain not containing the origin. The branches fit together in a simple way; in general, we can get from one branch to another by moving around the origin a sufficient number of times, while varying the choice of $\log z$ continuously. We say that the branches form "analytic continuations" of each other.

We can further verify that for each branch of $\log z$, the rule

$$\frac{d}{dz}\log z = \frac{1}{z} \tag{9–39}$$

remains valid. The familiar identities are also satisfied (Problems 4 and 5 below).

The *general exponential function* a^z is defined, for $a \neq 0$, by the equation

$$a^z = e^{z \log a} = \exp(z \log a). \tag{9–40}$$

Thus for $z = 0$, $a^0 = 1$. Otherwise, $\log a = \log |a| + i \arg a$, and we obtain many values: $a^z = \exp[z(\log |a| + i(\alpha + 2n\pi))]$, $(n = 0, \pm 1, \pm 2, \ldots)$, where α denotes one choice of $\arg a$. For example,

$$(1+i)^i = \exp\left[i\left\{\log\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)\right\}\right]$$
$$= e^{-(\pi/4)-2n\pi}(\cos\log\sqrt{2} + i\sin\log\sqrt{2}).$$

If z is a positive integer m, a^z reduces to a^m and has only one value. The same holds for $z = -m$, and we have

$$a^{-m} = \frac{1}{a^m}. \tag{9–41}$$

If z is a fraction p/q (in lowest terms), we find that a^z has q distinct values which are the qth roots of a^p. (See Eq. (0–14).)

If a fixed choice of $\log a$ is made in (9–40), then a^z is simply e^{cz}, $c = \log a$ and is hence an analytic function of z for all z. Each choice of $\log a$ determines such a function.

If a and z are interchanged in (9–40), we obtain the *general power function*,

$$z^a = e^{a \log z}. \tag{9–42}$$

If an analytic branch of $\log z$ is chosen as above, then this function becomes an analytic function of an analytic function and is hence analytic in the domain chosen. In particular, the *principal value* of z^a is defined as the analytic function $z^a = e^{a \operatorname{Log} z}$, in terms of the principal value of $\log z$.

For example, if $a = \frac{1}{2}$, we have

$$z^{1/2} = e^{(1/2)\log z} = e^{(1/2)(\log r + i\theta)} = e^{(1/2)\log r}\, e^{(1/2)i\theta}$$
$$= \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right),$$

as in Eq. (0–14). If $\operatorname{Log} z$ is used, then $\sqrt{z} = f_1(z)$ becomes analytic in the domain of Fig. 9–5. A second analytic branch $f_2(z)$ in the same domain is obtained by requiring that $\pi < \theta < 3\pi$. These are the only two analytic branches which can be obtained in this domain. It should be remarked that these two branches are related by the equation $f_2(z) = -f_1(z)$. For f_2 is obtained from f_1 by increasing θ by 2π, which replaces $e^{(1/2)i\theta}$ by

$$e^{(1/2)i(\theta+2\pi)} = e^{\pi i} e^{(1/2)i\theta} = -e^{(1/2)i\theta}.$$

The functions $\sin^{-1} z$ and $\cos^{-1} z$ are defined as the inverses of $\sin z$ and $\cos z$. We then find

$$\sin^{-1} z = \frac{1}{i} \log [iz \pm \sqrt{1 - z^2}],$$
$$\cos^{-1} z = \frac{1}{i} \log [z \pm i\sqrt{1 - z^2}]. \tag{9-43}$$

The proofs are left to the exercises (Problem 2). It can be shown that analytic branches of both these functions can be defined in each simply-connected domain not containing the points ± 1. For each z other than ± 1, one has two choices of $\sqrt{1 - z^2}$ and then an infinite sequence of choices of the logarithm, differing by multiples of $2\pi i$.

Problems

1. Obtain all values of each of the following:

(a) $\log 2$ (b) $\log i$ (c) $\log (1 - i)$ (d) i^i (e) $(1 + i)^{2/3}$ (f) $i^{\sqrt{2}}$ (g) $\sin^{-1} 1$ (h) $\cos^{-1} 2$

2. Prove the formulas (9–43). [*Hint:* If $w = \sin^{-1} z$, then $2iz = e^{iw} - e^{-iw}$; multiply by e^{iw} and solve the resulting equation as a quadratic for e^{iw}.]

3. (a) Evaluate $\sin^{-1} 0$, $\cos^{-1} 0$.
 (b) Find all roots of $\sin z$ and $\cos z$ [compare part (a)].

4. Show that each branch of $\log z$ is analytic in each domain in which θ varies continuously and that

$$(d/dz) \log z = 1/z.$$

[*Hint:* Show from the equations $x = r \cos \theta$, $y = r \sin \theta$ that $\partial\theta/\partial x = -y/r^2$, $\partial\theta/\partial y = x/r^2$. Show that the Cauchy-Riemann equations hold for $u = \log r$, $v = \theta$.]

5. Prove the following identities in the sense that, for proper selection of values of the multiple-valued functions concerned, the equation is correct for each allowed choice of the variables:

(a) $\log (z_1 \cdot z_2) = \log z_1 + \log z_2 \quad (z_1 \neq 0,\ z_2 \neq 0)$
(b) $e^{\log z} = z \quad (z \neq 0)$
(c) $\log e^z = z$
(d) $\log z_1^{z_2} = z_2 \log z_1 \quad (z_1 \neq 0)$

6. For each of the following determine all analytic branches of the multiple-valued function in the domain given:

(a) $\log z$, $x < 0$ (b) $\sqrt[3]{z}$, $x > 0$

7. Prove that for the analytic function z^a (principal value),

$$(d/dz)z^a = (az^a)/z = az^{a-1}.$$

8. Plot the functions $u = \operatorname{Re}(\sqrt{z})$ and $v = \operatorname{Im}(\sqrt{z})$ as functions of x and y and show the two branches described in the text.

Answers

1. (a) $0.693 + 2n\pi i$ (b) $i(\frac{1}{2}\pi + 2n\pi)$ (c) $0.347 + i(\frac{7}{4}\pi + 2n\pi)$
(d) $\exp(-\frac{1}{2}\pi - 2n\pi)$ (e) $\sqrt[3]{2} \exp\left(\frac{1}{6}\pi i + \frac{4n\pi}{3} i\right)$
(f) $\exp\left(\frac{\sqrt{2}}{2} \pi i + 2\sqrt{2}\, n\pi i\right)$ (g) $\frac{1}{2}\pi + 2n\pi$ (h) $2n\pi \pm 1.317i$

The range of n is $0, \pm 1, \pm 2, \ldots$, except in (e), where it is 0, 1, 2.

3. (a) and (b) $n\pi$ and $(\pi/2) + n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$
6. (a) $\log r + i\theta$, $\frac{1}{2}\pi + 2n\pi < \theta < \frac{3}{2}\pi + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$
 (b) $\sqrt[3]{r} \exp(i\theta/3)$, $-(\pi/2) + 2n\pi < \theta < (\pi/2) + 2n\pi \quad (n = 0, 1, 2)$

9–8 Integrals of analytic functions. Cauchy integral theorem. All paths in the integrals concerned here, as elsewhere in the chapter, are assumed to be piecewise smooth.

The following theorem is fundamental for the theory of analytic functions:

THEOREM 12 (Cauchy integral theorem). *If $f(z)$ is analytic in a simply-connected domain D, then*

$$\oint_C f(z)\, dz = 0$$

on every simple closed path C in D (Fig. 9–6).

Proof. We have, by (9–22) above,

$$\oint_C f(z)\, dz = \oint_C u\, dx - v\, dy + i \oint_C v\, dx + u\, dy.$$

The two real integrals are equal to zero (see Section 5–6 above)

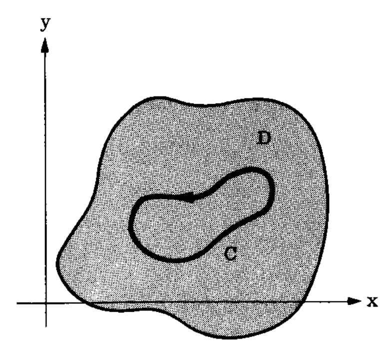

FIG. 9–6. Cauchy integral theorem.

provided u and v have continuous derivatives in D and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$$

These are just the Cauchy-Riemann equations. Hence

$$\oint_C f(z)\,dz = 0 + i\cdot 0 = 0.$$

This theorem can be stated in an equivalent form:

THEOREM 12′. *If $f(z)$ is analytic in the simply-connected domain D, then $\int f(z)\,dz$ is independent of the path in D.*

For independence of path and equaling zero on closed paths are equivalent properties of line integrals. If C is a path from z_1 to z_2, we can now write

$$\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz,$$

the integral being the same for all paths C from z_1 to z_2.

THEOREM 13. *Let $f(z) = u + iv$ be defined in domain D and let u and v have continuous partial derivatives in D. If*

$$\oint_C f(z)\,dz = 0 \tag{9–44}$$

on every simple closed path C in D, then $f(z)$ is analytic in D.

Proof. The condition (9–44) implies that

$$\oint_C u\,dx - v\,dy = 0, \qquad \oint_C v\,dx + u\,dy = 0$$

on all simple closed paths C; that is, the two real line integrals are independent of path in D. Therefore, by Theorem III in Section 5–6,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x};$$

since the Cauchy-Riemann equations hold, f is analytic.

This theorem can be proved with the assumption that u and v have continuous derivatives in D replaced by the assumption that f is continuous in D; it is then known as *Morera's* theorem. For a proof, see Chapter 5 of Vol. I of the book by Knopp listed at the end of the chapter.

THEOREM 14. *If $f(z)$ is analytic in D, then*

$$\int_{z_1}^{z_2} f'(z)\,dz = f(z)\Big|_{z_1}^{z_2} = f(z_2) - f(z_1) \tag{9–45}$$

on every path in D from z_1 to z_2. In particular,

$$\oint f'(z)\,dz = 0$$

on every closed path in D.

Proof. By (9–29) above,

$$\begin{aligned}\int_{z_1}^{z_2} f'(z)\,dz &= \int_{z_1}^{z_2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(dx + i\,dy)\\ &= \int_{z_1}^{z_2}\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + i\int_{z_1}^{z_2}\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\\ &= \int_{z_1}^{z_2} du + i\,dv = (u + iv)\Big|_{z_1}^{z_2} = f(z_2) - f(z_1).\end{aligned}$$

This rule is the basis for evaluation of simple integrals, just as in elementary calculus. Thus we have

$$\int_i^{1+i} z^2\,dz = \frac{z^3}{3}\Big|_i^{1+i} = \frac{(1+i)^3 - i^3}{3} = -\tfrac{2}{3} + i,$$

$$\int_i^{-i}\frac{1}{z^2}\,dz = -\frac{1}{z}\Big|_i^{-i} = -i - i = -2i.$$

In the first of these any path can be used; in the second, any path not through the origin.

THEOREM 15. *If $f(z)$ is analytic in D and D is simply-connected, then*

$$F(z) = \int_{z_1}^{z} f(z)\,dz \quad (z_1 \text{ fixed in } D) \tag{9–46}$$

is an indefinite integral of $f(z)$; that is, $F'(z) = f(z)$. Thus $F(z)$ is itself analytic.

Proof. Since $f(z)$ is analytic in D and D is simply-connected, $\int_{z_1}^{z} f(z)\,dz$ is independent of path and defines a function F which depends only on the upper limit z. We have, further, $F = U + iV$, where

$$U = \int_{z_1}^{z} u\,dx - v\,dy, \qquad V = \int_{z_1}^{z} v\,dx + u\,dy$$

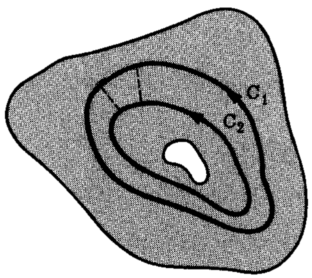

FIG. 9–7. Cauchy theorem for doubly-connected domain.

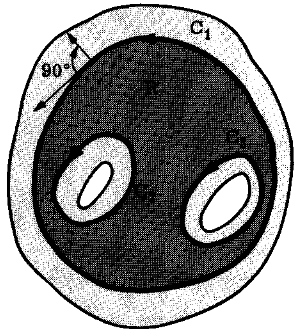

FIG. 9–8. Cauchy theorem for triply-connected domain.

and both integrals are independent of path. Hence $dU = u\,dx - v\,dy$, $dV = v\,dx + u\,dy$. Thus U and V satisfy the Cauchy-Riemann equations, so that $F = U + iV$ is analytic and

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z).$$

Cauchy's theorem for multiply-connected domains. If $f(z)$ is analytic in a multiply-connected domain D, then we cannot conclude that

$$\oint f(z)\,dz = 0$$

on every simple closed path C in D. Thus, if D is the doubly-connected domain of Fig. 9–7 and C is the curve C_1 shown, then the integral around C need not be zero. However, by introducing cuts, we can reason that

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz; \tag{9–47}$$

that is, the integral has the same value on all paths which go around the inner "hole" once in the positive direction. For a triply-connected domain, as in Fig. 9–8, we obtain the equation

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz. \tag{9–48}$$

This can be written in the form

$$\oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz = 0; \tag{9–49}$$

Eq. (9–49) states that the integral around the complete boundary of a certain region in D is equal to zero. More generally, we have the following theorem:

THEOREM 16 (Cauchy's theorem for multiply-connected domains). *Let $f(z)$ be analytic in a domain D and let $C_1, \ldots, C_n$ be n simple closed curves in D which together form the boundary B of a region R contained in D. Then*

$$\int_B f(z)\,dz = 0,$$

where the direction of integration on B is such that the outer normal is 90° behind the tangent vector in the direction of integration.

9–9 Cauchy's integral formula. Now let D be a simply-connected domain and let z_0 be a fixed point of D. If $f(z)$ is analytic in D, the function $f(z)/(z - z_0)$ will fail to be analytic at z_0. Hence

$$\oint \frac{f(z)}{z - z_0}\,dz$$

will in general not be zero on a path C enclosing z_0. However, as above, this integral will have the same value on all paths C about z_0. To determine this value, we reason that if C is a very small circle of radius R about z_0, then $f(z_0)$ has, by continuity, approximately the constant value $f(z_0)$ on the path. This suggests that

$$\oint_C \frac{f(z)}{z - z_0}\,dz = f(z_0)\cdot \oint_{|z-z_0|=R} \frac{dz}{z - z_0} = f(z_0)\cdot 2\pi i,$$

since we find

$$\oint_{|z-z_0|=R} \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{Rie^{i\theta}}{Re^{i\theta}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i,$$

with the aid of the substitution: $z - z_0 = Re^{i\theta}$. The correctness of the conclusion reached is the content of the following fundamental result:

THEOREM 17 (Cauchy integral formula). *Let $f(z)$ be analytic in a domain D. Let C be a simple closed curve in D, within which $f(z)$ is analytic and let z_0 be inside C. Then*

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz. \tag{9–50}$$

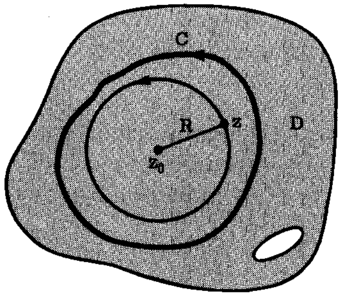

FIG. 9–9. Cauchy integral formula.

Proof. The domain D is not required to be simply-connected, but since f is analytic within C, the theorem concerns only a simply-connected part of D, as shown in Fig. 9–9. We reason as above to conclude that

$$\oint_C \frac{f(z)}{z - z_0}\,dz = \oint_{|z-z_0|=R} \frac{f(z)}{z - z_0}\,dz.$$

It remains to show that the integral on the right is indeed $f(z_0) \cdot 2\pi i$. Now, since $f(z_0) = \text{const}$,

$$\oint \frac{f(z_0)}{z - z_0}\,dz = f(z_0) \oint \frac{dz}{z - z_0} = f(z_0) \cdot 2\pi i,$$

where we integrate always on the circle $|z - z_0| = R$. Hence, on the same path,

$$\oint \frac{f(z)}{z - z_0}\,dz - f(z_0) \cdot 2\pi i = \oint \frac{f(z) - f(z_0)}{z - z_0}\,dz. \qquad (9\text{–}51)$$

Now $|z - z_0| = R$ on the path, and since $f(z)$ is continuous at z_0, $|f(z) - f(z_0)| < \epsilon$ for $R < \delta$, for each preassigned $\epsilon > 0$. Hence, by Theorem 7,

$$\left|\oint \frac{f(z) - f(z_0)}{z - z_0}\,dz\right| < \frac{\epsilon}{R} \cdot 2\pi R = 2\pi\epsilon.$$

Thus the absolute value of the integral can be made as small as desired by choosing R sufficiently small. But the integral has the same value for all choices of R. This is possible only if the integral is zero for all R. Hence the left side of (9–51) is zero and (9–50) follows.

The integral formula (9–50) is remarkable in that it expresses the values of the function $f(z)$ at points z_0 inside the curve C in terms of the values

along C alone. If C is taken as a circle $z = z_0 + Re^{i\theta}$, then (9–50) reduces to the following:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta. \qquad (9\text{–}52)$$

Thus the *value of an analytic function at the center of a circle equals the average (arithmetic mean) of the values on the circumference.*

Just as with the Cauchy integral theorem, the Cauchy integral formula can be extended to multiply-connected domains. Under the hypotheses of Theorem 16,

$$f(z_0) = \frac{1}{2\pi i} \int_B \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi i}\left(\oint_{C_1} \frac{f(z)}{z - z_0}\,dz + \oint_{C_2} \frac{f(z)}{z - z_0}\,dz + \cdots\right), \qquad (9\text{–}53)$$

where z_0 is any point inside the region R bounded by C_1 (the outer boundary), $C_2, \ldots, C_n$. The proof is left as an exercise (Problem 6 below).

Problems

1. Evaluate the following integrals:

(a) $\oint z^2 \sin z\,dz$ on the ellipse $x^2 + 2y^2 = 1$

(b) $\oint \frac{z^2}{z+1}\,dz$ on the circle $|z - 2| = 1$

(c) $\int_1^{2i} ze^z\,dz$ on the line segment joining the endpoints

(d) $\int_{1+i}^{1-i} \frac{1}{z^2}\,dz$ on the parabola $2y^2 = x + 1$

2. (a) Evaluate $\int_{-i}^{i} (dz/z)$ on the path $z = e^{it}$, $-\pi/2 \leqq t \leqq \pi/2$, with the aid of the relation $(\log z)' = 1/z$, for an appropriate branch of $\log z$.

(b) Evaluate $\int_i^{-i} (dz/z)$ on the path $z = e^{it}$, $\pi/2 \leqq t \leqq 3\pi/2$, as in part (a).

(c) Why does the relation $(\log z)' = 1/z$ not imply that the sum of the two integrals of parts (a) and (b) is zero?

3. A certain function $f(z)$ is known to be analytic except for $z = 1$, $z = 2$, $z = 3$, and it is known that

$$\oint_{C_k} f(z)\,dz = a_k \quad (k = 1, 2, 3),$$

where C_k is a circle of radius $\frac{1}{2}$ with center at $z = k$. Evaluate

$$\oint f(z)\, dz$$

on each of the following paths:

(a) $|z| = 4$ (b) $|z| = 2.5$ (c) $|z - 2.5| = 1$

4. A certain function $f(z)$ is analytic except for $z = 0$, and it is known that

$$\lim_{z\to\infty} zf(z) = 0.$$

Show that

$$\oint f(z)\, dz = 0$$

on every simple closed path not passing through the origin. [*Hint:* Show that the value of the integral on a path $|z| = R$ can be made as small as desired by making R sufficiently large.]

5. Evaluate each of the following with the aid of the Cauchy integral formula:

* (a) $\displaystyle\oint \frac{z}{z-3}\, dz$ on $|z| = 5$ * (b) $\displaystyle\oint \frac{e^z}{z^2 - 3z}\, dz$ on $|z| = 1$

* (c) $\displaystyle\oint \frac{z+2}{z^2-1}\, dz$ on $|z| = 2$ * (d) $\displaystyle\oint \frac{\sin z}{z^2+1}\, dz$ on $|z| = 2$

[*Hint for* (c) *and* (d): Expand the rational function in partial fractions.]

6. Prove (9–53) under the hypotheses stated.

7. Prove that if $f(z)$ is analytic in domain D and $f'(z) \equiv 0$, then $f(z) \equiv$ constant. [*Hint:* Apply Theorem 14.]

Answers

1. (a) 0 (b) 0 (c) $(2i - 1)e^{2i}$ (d) $-i$
2. (a) πi (b) πi
3. (a) $a_1 + a_2 + a_3$ (b) $a_1 + a_2$ (c) $a_2 + a_3$
5. (a) $6\pi i$ (b) $-2\pi i/3$ (c) $2\pi i$ (d) $2\pi i \sinh 1$