Frictional Heating Along A Fault

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During an earthquake, one of the ways energy is released is through work against friction on the fault. This work is converted to heat, which can be described by a diffusion equation.

If we assume that the fault is an infinite bar, this equation will be one dimensional, where the variables are as follows:

- $T$: Temperature
- $Q$: Rate of Frictional Heat generated in fault zone
- $\rho$: Density
- $\kappa$: Thermal Diffusivity
- $y$: Fault Coordinate
- $c$: Heat capacity
- $\tau$: Time of Faulting
The Problem
The Governing Heat Transfer Equation:

\[ \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial y^2} + \frac{Q}{c \rho} \]  \hspace{1cm} (1)

Assume forcing term Q is a delta function for now.

\[ \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial y^2} + \delta(y - \zeta) \]

Take Fourier Transform of both sides.

\[ \frac{d \hat{T}}{dt} = -\kappa k^2 \hat{T} + \frac{1}{\sqrt{2\pi}} e^{-ik\zeta} \delta(t - \tau) \]

This becomes an inhomogeneous, linear ordinary differential equation.
SOLVE ODE

\[ \frac{dT}{dt} = -\kappa k^2 T + \frac{1}{\sqrt{2\pi}} e^{-ik\zeta \delta (t - \tau)} \]

Solve ODE of the form

\[ y'(x) + p_o(x)y(x) = f(x) \]

Where:

\[ p_o(x) = \kappa k^2 \quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-ik\zeta \delta (t - \tau)} \]

Solve with integration factor:

\[ I(x) = e^{\int \kappa k^2 dt} = e^{\kappa k^2 t} \]

Then multiply integration factor, solution will be of the form:

\[ \hat{T} = \frac{c_1}{I(x)} + \frac{1}{I(x)} \int f(t)I(t)dt \]
With the boundary condition, $c_1$ is 0, so substituting:

$$0 + e^{-\kappa k^2 t} \int e^{\kappa k^2 t} \left( \frac{1}{\sqrt{2\pi}} e^{-ik\zeta} \delta(t-\tau) \right) dt$$

And by integration, obtain:

$$= \frac{e^{\kappa k^2 t}}{\sqrt{2\pi}} \left( e^{\kappa k^2 \tau - ik\zeta} \right) = \frac{e^{\kappa k^2 \tau - \kappa k^2 t - ik\zeta}}{\sqrt{2\pi}}$$

$$= \frac{e^{-\kappa k^2 (t-\tau) - ik\zeta}}{\sqrt{2\pi}} (\sigma(t-\tau)) = \hat{T}$$
The inverse Fourier Transform of this solution will be used as the $G$, the Green’s Function.

\[
G(y - \zeta, t - \tau) = \frac{\sigma(t - \tau)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iky} \hat{T}(k, t) dk
\]

\[
= \frac{\sigma(t - \tau)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iky} e^{-\kappa k^2 (t - \tau) - i k \zeta} dk
\]

\[
= \frac{\sigma(t - \tau)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(y - \zeta) - \kappa k^2 (t - \tau)} dk
\]

This is a Gaussian Function.
The Fourier Transform of a Gaussian is as follows:

$$\mathcal{F}[ae^{-bk^2}] = \frac{a}{\sqrt{2b}} \exp\left[-\frac{m^2}{4b}\right]$$

Let $a = 1$, $b = \kappa(t-\tau)$. Take:

$$\mathcal{F}[e^{\kappa k^2(t-\tau)}] = \frac{1}{\sqrt{2\kappa(t-\tau)}} \exp\left[-\frac{(y-\zeta)^2}{4\kappa(t-\tau)}\right]$$

So using this in $G$:

$$G = \frac{\sigma(t-\tau)}{\sqrt{(2\pi)(2\kappa(t-\tau))}} \exp\left[-\frac{(y-\zeta)^2}{4\kappa(t-\tau)}\right]$$

$$= \frac{\sigma(t-\tau)}{2\sqrt{\pi\kappa(t-\tau)}} \exp\left[-\frac{(y-\zeta)^2}{4\kappa(t-\tau)}\right]$$
The solution for $T$ is then given by the superposition integral:

$$T(y, t) = \int_0^t \int_{-\infty}^\infty G(y - \zeta) (t - \tau) p(\zeta, \tau) \, d\zeta \, d\tau$$

If we replace our forcing term from earlier, we obtain the solution:

$$p(\zeta, \tau) = \frac{Q(\zeta, \tau)}{c \rho}$$

$$\Delta T(y, t) = \int_0^t \int_{-\infty}^\infty \frac{\sigma(t - \tau)}{2 \sqrt{\pi \kappa (t - \tau)}} \exp \left[ -\frac{(y - \zeta)^2}{4 \kappa (t - \tau)} \right] \frac{Q(\zeta, \tau)}{c \rho} \, d\zeta \, d\tau$$

$$\Delta T(y, t) = \frac{\sigma(t - \tau)}{2 \sqrt{\pi \kappa}} \int_0^t \int_{-\infty}^\infty \exp \left[ -\frac{(y - \zeta)^2}{4 \kappa (t - \tau)} \right] \frac{Q(\zeta, \tau)}{c \rho \sqrt{t - \tau}} \, d\zeta \, d\tau$$
An expression for $Q$:

If $|y| < w$, and $t > 0$ then

$$Q(y, t) = \frac{\sigma_f}{2w} \frac{\partial D}{\partial t}$$

If $|y| > w$, then $Q(y, t) = 0$

$$\Delta T(y, t) = \frac{\sigma (t - \tau)}{2c \rho \sqrt{\pi \kappa}} \int_0^t \int_{-\infty}^\infty \exp \left[ -\frac{(y - \zeta)^2}{4\kappa(t - \tau)} \right] \frac{\sigma_f}{2w} \frac{1}{\sqrt{t - \tau}} d\zeta d\tau$$

Use an error function to solve the integral.

Let:

$$-\phi = \left( \frac{y - \zeta}{\sqrt{4\kappa(t - \tau)}} \right) = \left( \frac{y - \zeta}{2\sqrt{\kappa(t - \tau)}} \right)$$
Then:

$$\Delta T(y, t) = \int_0^t \frac{\sigma_f}{2w} \frac{\partial D}{\partial t} \left( \int_0^w e^{-\phi^2} d\phi - \int_0^{-w} e^{-\phi^2} d\phi \right)$$

$$\Delta T(y, t) = \frac{\sigma_f}{4c\rho w} \int_0^t \left( \text{erf} \left[ \frac{y + w}{2\sqrt{2\kappa(t - \tau)}} \right] - \text{erf} \left[ \frac{y - w}{2\sqrt{2\kappa(t - \tau)}} \right] \right) \frac{\partial D}{\partial \tau} \sigma_f d\tau$$
How do we solve this?

\[ \Delta T(y, t) = \frac{\sigma_f}{4 \lambda \rho w} \int_0^t \left( \text{erf} \left( y + \frac{w}{2 \sqrt{2 \kappa (t - \tau)}} \right) - \text{erf} \left( y - \frac{w}{2 \sqrt{\kappa (t - \tau)}} \right) \right) \frac{\partial D}{\partial \tau} \sigma_f d\tau \]

Rectangle rule

Trapezium rule

Simpson’s rule
GRANITE, $\rho=2.75\text{g/cm}^3$, $k=3.2\text{Wm}^{-1}\text{K}^{-1}$
Why does this matter?
Pseudotachylite is a fault rock that has the appearance of the basaltic glass, tachylyte. It is dark in color and has a glassy appearance. However, the glass has normally been completely devitrified into very fine-grained material with radial and concentric clusters of crystals. It may contain clasts of the country rock and occasionally crystals with quench textures that began to crystallize from the melt.

Maddock, R.H., 1983
Other results

Fialko & Khazan, 2005
Viscous breaking vs. fault lubrication


The transition occurs if:

- The driving stress is sufficiently high to overcome viscous drag.
- There is no transfer of strain off of the fused fault surface.
- The time scale of the thermal runaway is small compared to the duration of seismic slip.
Final Remarks

- The thermal behavior shown here is only for a unidimensional crack. No rupture or healing front is assumed.

- We assume all of the frictional energy becomes heat energy (in reality only 95% of it does so).

- The seismic source cannot be properly modeled as merely a mechanical problem.