APPLICATIONS AND REVIEW OF FOURIER TRANSFORM/SERIES

(Reference – *The Fourier Transform and its Application, second edition*, R.N. Bracewell, McGraw-Hill Book Co., New York, 1978.)

It may seem unusual that we begin a course on Geodynamics by reviewing fourier transforms and fourier series. However, you will see that fourier analysis is used in almost every aspect of the subject ranging from solving linear differential equations to developing computer models, to the processing and analysis of data. We won't be using fourier analysis for the first few lectures, but I'll introduce the concepts today so that people who are less familiar with the topic can have time for review. In the first few lectures, I'll also discuss plate tectonics. I imagine that students with physics and math backgrounds may have to spend some time reviewing plate tectonics. Hopefully everyone will be busy the first two weeks. Next class I'll give a homework assignment involving fourier transforms. Later we'll have a short quiz on plate tectonics.

Some applications of fourier transforms

Solving linear partial differential equations (PDE's):

Gravity/magnetics	Laplace	$\nabla^2\Phi = 0$
Elasticity (flexure)	Biharmonic	$\nabla^4\Phi = 0$
Heat Conduction	Diffusion	$\nabla^2\Phi - \delta\,\Phi/\,\delta t = 0$
Wave Propagation	Wave	$\nabla^2\Phi - \delta^2\Phi/\,\delta t^2 = 0$

Designing and using antennas:

Seismic arrays and streamers
Multibeam echo sounder and side scan sonar
Interferometers – VLBI – GPS
Synthetic Aperture Radar (SAR) and Interferometric SAR (InSAR)

Image Processing and filters:

Transformation, representation, and encoding
Smoothing and sharpening
Restoration, blur removal, and Wiener filter

Data Processing and Analysis:

High-pass, low-pass, and band-pass filters
Cross correlation – transfer functions – coherence
Signal and noise estimation – encoding time series

Definitions of fourier transforms

The 1-dimensional fourier transform is defined as:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi kx} dx \qquad F(k) = \Im\left[f(x)\right] \quad \text{- forward transform}$$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{i2\pi kx} dk \qquad f(x) = \Im^{-1}\left[F(k)\right] \quad \text{- inverse transform}$$

where x is distance and k is wavenumber where $k = 1/\lambda$ and λ is wavelength. These equations are more commonly written in terms of time t and frequency ν where $\nu = 1/T$ and T is the period. The 2-dimensional fourier transform is defined as:

$$F(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x}) e^{-i2\pi(\mathbf{k}\cdot\mathbf{x})} d^2\mathbf{x} \qquad F(\mathbf{k}) = \Im_2\left[f(\mathbf{x})\right]$$

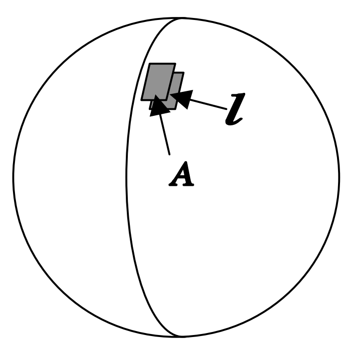
$$f(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mathbf{k}) e^{i2\pi(\mathbf{k}\cdot\mathbf{x})} d^2\mathbf{k} \qquad f(\mathbf{x}) = \Im_2^{-1}\left[F(\mathbf{k})\right]$$

where $\mathbf{x} = (x, y)$ is the position vector, $\mathbf{k} = (k_x, k_y)$ is the wavenumber vector, and $(\mathbf{k} \cdot \mathbf{x}) = k_x x + k_y y$.

Why use fourier transforms on a nearly spherical earth?

If you have taken geomagnetism or global seismology, you were taught to expand a function of latitude and longitude in spherical harmonics. Later in the course we will also use spherical harmonics to represent large-scale variations in the gravity field and to represent viscous mantle flow. However, throughout the course we will be dealing with problems related to the crust and lithosphere. In these cases a flat-earth approximation is both adequate and practical for the following reasons:

A. Cartesian geometry is a good approximation. Consider a small patch of crust or lithosphere on the surface of a sphere. If the area of the patch is A is much less than the area of the earth and the thickness l of the patch is much less than the radius of the earth R_e, then the Cartesian geometry will be adequate.



$$A << 4\pi R_e^2$$
$$l << R_e$$

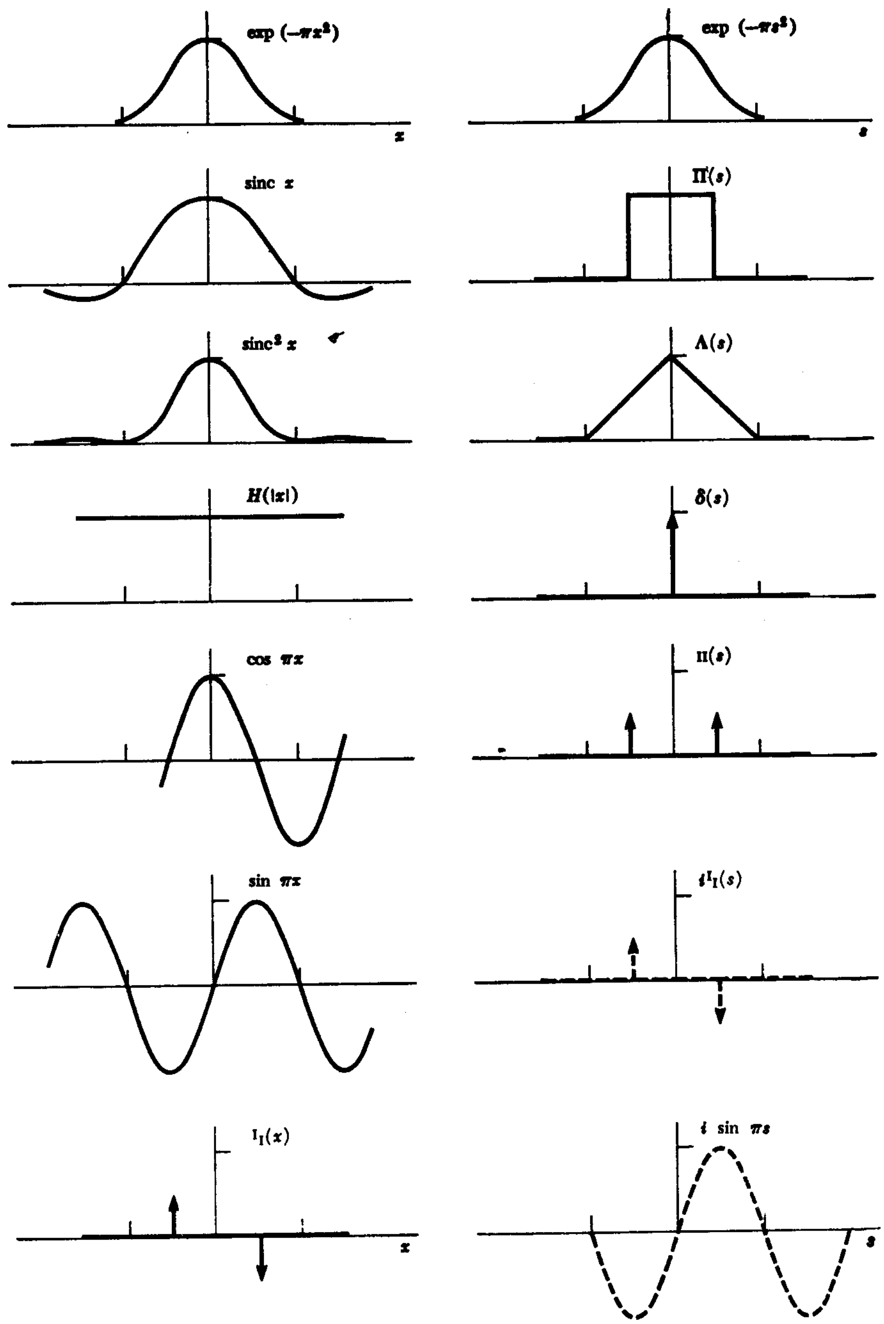
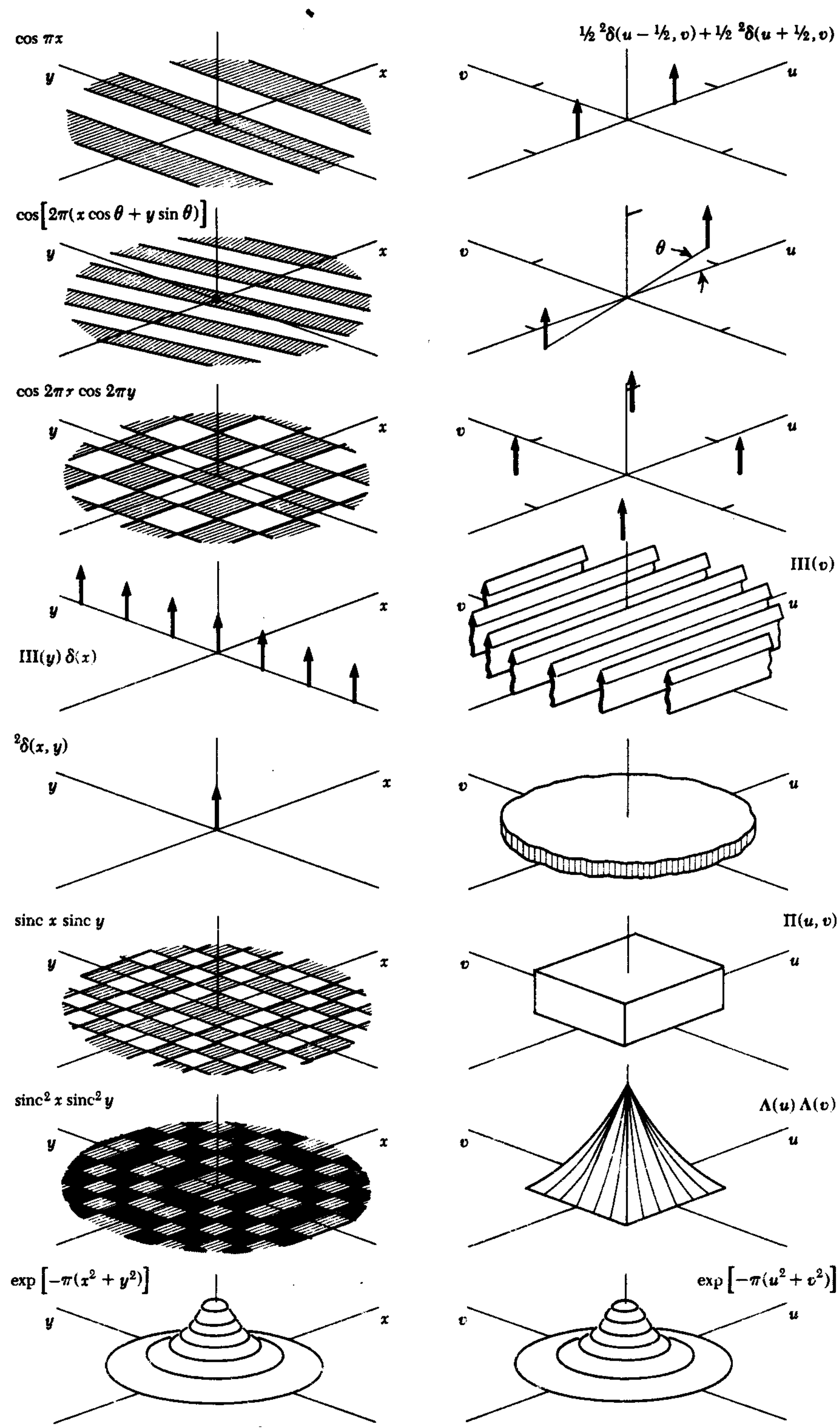
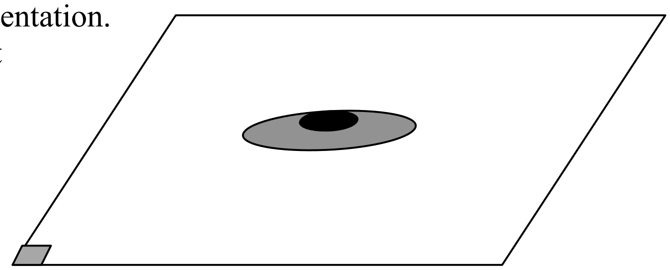


Fig. 6.1 Some Fourier transform pairs for reference.


$\cos \pi x$
$\frac{1}{2}\,{}^2\delta(u - \frac{1}{2}, v) + \frac{1}{2}\,{}^2\delta(u + \frac{1}{2}, v)$
$\cos\left[2\pi(x \cos\theta + y \sin\theta)\right]$
θ
$\cos 2\pi x \cos 2\pi y$
$\mathrm{III}(v)$
$\mathrm{III}(y)\,\delta(x)$
${}^2\delta(x, y)$
$\mathrm{sinc}\, x\, \mathrm{sinc}\, y$
$\Pi(u, v)$
$\mathrm{sinc}^2 x\, \mathrm{sinc}^2 y$
$\Lambda(u)\,\Lambda(v)$
$\exp\left[-\pi(x^2 + y^2)\right]$
$\exp\left[-\pi(u^2 + v^2)\right]$
x
y
u
v

In this case the patch is nearly flat and it is also quite thin. The area at the bottom of the patch is about equal to the area at the top of the patch so a 1-D approximation or a plane-stress approximation may be adequate when we solve the heat conduction or flexure equations, respectively. Note that on a planet like Mars where the lithospheric thickness is a large fraction of the radius, the use of the Cartesian geometry may not be appropriate.

B. Cartesian geometry is practical. Consider the representation of a function on a spherical earth. Suppose we want a spherical harmonic representation of the patch of seafloor illustrated below which contains a seamount (100 km diameter). The depth sampling must be better than 4 km by 4 km for adequate representation. Since the circumference of the Earth is about 40,000 km, the maximum spherical harmonic degree l_{max} must be at least 10^4 and thus 10^8 coefficients will be needed. Clearly this will be impractical from a computational standpoint and, moreover, most of the surface will have the same depth so most of the coefficients do not contain useful information. Since the seamount has a diameter of about 100 km, we can work with a smaller patch of dimensions 400 km by 400 km. If a fourier representation is used, only $(400/4)^2 = 10^4$ coefficients will be needed.



Fourier sine and cosine transforms

Any function $f(x)$ can be decomposed into odd $O(x)$ and even $E(x)$ components.

$$f(x) = E(x) + O(x)$$

$$E(x) = \frac{1}{2}\left[f(x) + f(-x)\right] \qquad O(x) = \frac{1}{2}\left[f(x) - f(-x)\right]$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi kx} dx$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$F(k) = \int_{-\infty}^{\infty} f(x)\cos(2\pi kx)dx - i\int_{-\infty}^{\infty} f(x)\sin(2\pi kx)dx$$

odd part cancels even part cancels

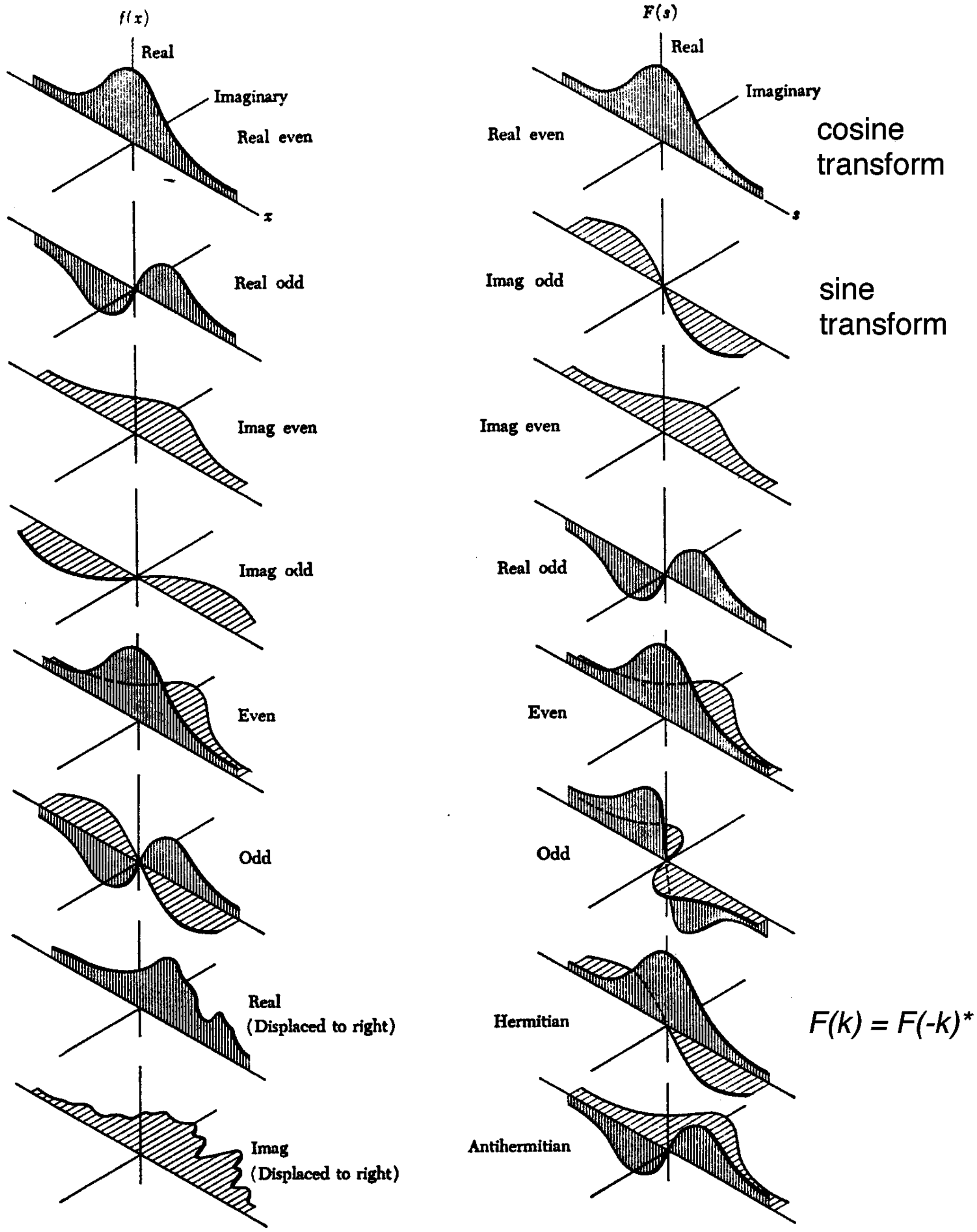


Fig. 2.5 Symmetry properties of a function and its Fourier transform.

(scan 600DPI line art)

$$F(k) = 2\int_{o}^{\infty} E(x)\cos(2\pi kx)dx - 2i\int_{o}^{\infty} O(x)\sin(2\pi kx)dx$$

cosine transform   sine transform

You have probably seen fourier cosine and sine transforms, but it is better to use the complex exponential form.

Properties of fourier transforms

The following are some important properties of fourier transforms that you should derive for yourself at least once. You'll find derivations in Bracewell. Once you have derived and understand these properties, you can treat them as tools. Very complicated problems can be simplified using these tools. For example, when solving a linear partial differential equation, one uses the derivative property to reduce the differential equation to an algebraic equation.

similarity property $\Im[f(ax)] = \frac{1}{|a|}F\left(\frac{k}{a}\right)$

shift property $\Im[f(x - a)] = e^{-i2\pi ka}F(k)$

differentiation property $\Im\left[\frac{df}{dx}\right] = i2\pi k\, F(k)$

convolution property $\Im\left[\int_{-\infty}^{\infty} f(u)g(x - u)du\right] = F(k)G(k)$

Rayleigh' s theorem $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk$

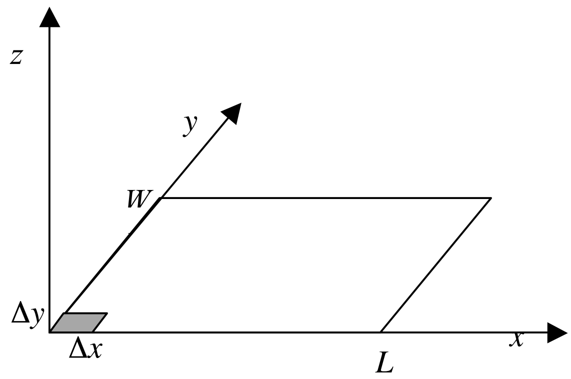
Note: These properties are equally valid in 2-dimensions or even n-dimensions. The properties also apply to discrete data. See Chapter 18 in Bracewell.

Fourier series

Many geophysical problems are concerned with a small area on the surface of the Earth.

W - width of area

L - length of area



The coefficients of the 2-dimensional Fourier series are computed by the following integration.

$$F_n^m = \frac{1}{LW}\int_o^L\int_o^W f(x,y)\exp\left[-i2\pi\left(\frac{m}{L}x+\frac{n}{W}y\right)\right]dydx$$

The function is reconstructed by the following summations over the fourier coefficients.

$$f(x,y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} F_n^m \exp\left[i2\pi\left(\frac{m}{L}x+\frac{n}{W}y\right)\right]$$

The finite size of the area leads to a discrete set of wavenumbers $k_x = m/L$, $k_y = n/W$ and a discrete set of fourier coefficients F_n^m. In addition to the finite size of the area, geophysical data commonly have a characteristic sampling interval Δx and Δy.

$I = L/\Delta x$ - number of points in the x-direction

$J = W/\Delta y$ - number of points in the y-direction

The Nyquist wavenumbers is $k_x = 1/(2\ \Delta x)$ and $k_y = 1/(2\ \Delta y)$ so there is a finite set of fourier coefficients $-I/2 < m < I/2$ and $-J/2 < n < J/2$. Recall the trapezoidal rule of integration.

$$\int_o^L f(x)dx \cong \sum_{i=0}^{I-1} f(x_i)\Delta x \quad \text{where } x_i = i\Delta x.$$

$$\int_o^L f(x)dx \cong \frac{L}{I}\sum_{i=0}^{I-1} f(x_i)$$

The discrete forward and inverse fourier transform are:

$$F_n^m = \frac{1}{IJ}\sum_{i=0}^{I-1}\sum_{j=0}^{J-1} f_i^j \exp\left[-i2\pi\left(\frac{m}{I}i+\frac{n}{J}j\right)\right]$$

The forward and inverse discrete fourier transforms are almost identical sums so one can use the same computer code for both operations.

$$f_{\mathrm{i}}^{\mathrm{j}} = \sum_{n=-I/2}^{I/2-1} \sum_{m=-J/2}^{J/2-1} F_n^m \exp\left[i2\pi \left(\frac{\mathrm{i}}{I} m + \frac{\mathrm{j}}{J} n \right) \right]$$

Sorry for the dual use of the symbol *i*. The *i* in front of the 2π is $\sqrt{-1}$ while the other non-itallic i's are integers.

Properties of fourier transforms

similarity $\mathcal{F}[f(ax)] = \frac{1}{|a|} F\left(\frac{k}{a}\right)$

$$\int_{-\infty}^{\infty} f(ax) e^{-i2\pi kx} dx \qquad \text{let } x' = ax \quad dx' = a\,dx \qquad k' = \frac{k}{a}$$

$$\int_{-\infty}^{\infty} f(x') e^{-i2\pi k' x'} \frac{1}{a} dx' = \frac{1}{a} F\left(\frac{k}{a}\right)$$

What happens when $a < 0$? (exercise)

shift $\mathcal{F}[f(x-a)] = e^{-i2\pi ka} F(k)$

$$\int_{-\infty}^{\infty} f(x-a) e^{-i2\pi kx} dx \qquad \text{let } x' = (x-a) \quad dx' = dx \qquad x = (x'+a)$$

$$\int_{-\infty}^{\infty} f(x') e^{-i2\pi k(x'+a)} dx' = e^{-i2\pi ka} F(k)$$

differentiation $\mathcal{F}\left[\frac{df}{dx}\right] = i2\pi k\, F(k)$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{i2\pi kx} dk \qquad \frac{df}{dx} = \int_{-\infty}^{\infty} i2\pi k\, F(k) e^{i2\pi kx} dk$$

$$\mathcal{F}\left[\frac{df}{dx}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i2\pi k' F(k') e^{i2\pi k' x} dk' \, e^{-i2\pi k x} dx$$

$$= \int_{-\infty}^{\infty} i2\pi k' F(k') \underbrace{\int_{-\infty}^{\infty} e^{i2\pi (k'-k)x} dx}_{\delta(k'-k)} dk'$$

$$= i2\pi k F(k)$$

<u>convolution</u> $\mathcal{F}\left[\int f(u) g(x-u) du\right] = F(k) G(k)$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u) g(x-u) du\right] e^{-i2\pi k x} dx$$

$$= \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g(x-u) e^{-i2\pi k x} dx\right] du$$

$$\overset{\shortparallel}{e^{-i2\pi k u} G(k)}$$

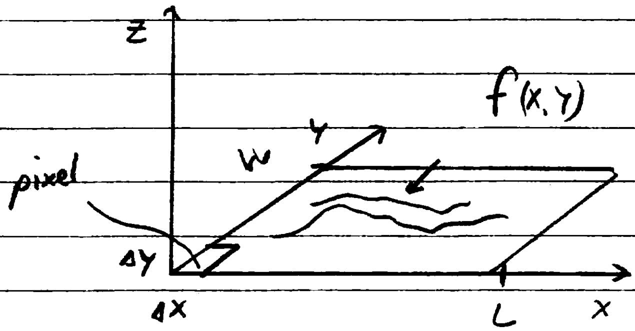
$$= G(k) \int_{-\infty}^{\infty} f(u) e^{-i2\pi k u} du = G(k) F(k)$$

skip deconvolution

+ Fourier Series

Fourier Series

W - width
L - length



fourier coefficients

$$F_n^m = \frac{1}{LW}\int_0^L\int_0^W f(x,y)\exp\left[-i2\pi\left(\frac{m}{L}x + \frac{n}{W}y\right)\right]dx\,dy$$

($\frac{m}{L}$ = k_x, $\frac{n}{W}$ = k_y)

the finite size of the area leads to discrete wavenumbers

$k_x = \frac{m}{L}$, $k_y = \frac{n}{W}$ and a discrete set

of fourier coefficients F_n^m

reconstruct original function

$$f(x,y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} F_n^m \exp\left[i2\pi\left(\frac{m}{L}x + \frac{n}{W}y\right)\right]$$

the sampling leads to further simplification

$I = \frac{L}{\Delta x}$ $J = \frac{W}{\Delta y}$

Nyquist wavenumber is $k_x = \frac{1}{2\Delta x}$ $k_y = \frac{1}{2\Delta y}$ so

there is a finite set of coefficients

$-\frac{I}{2} < m < \frac{I}{2}$ $-\frac{J}{2} < n < \frac{J}{2}$

Recall trapezoidal rule of integration

$$\int_0^L f(x)\,dx \cong \sum_{i=0}^{I-1} f(x_i)\,\Delta x \qquad \text{where } x_i = i\,\Delta x$$

$$\cong \frac{L}{I} \sum_{i=0}^{I-1} f(x_i)$$

The discrete forward and inverse transforms are

$$F_n^m = \frac{1}{IJ} \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} f_i^j \exp\left[-\sqrt{-1}\, 2\pi \left(\frac{m}{I} i + \frac{n}{J} j\right)\right]$$

$$f_i^j = \sum_{n=-\frac{I}{2}}^{\frac{I}{2}} \sum_{m=-\frac{J}{2}}^{\frac{J}{2}} F_n^m \exp\left[\sqrt{-1}\, 2\pi \left(\frac{i}{I} m + \frac{j}{J} n\right)\right]$$

These are the same formulas except for $\frac{1}{IJ}$