CHAPTER 2

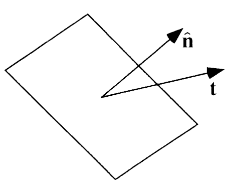
Stress and Strain

Any quantitative description of seismic wave propagation requires the ability to characterize the internal forces and deformations in solid materials. We now begin a brief review of those parts of stress and strain theory that will be needed in subsequent chapters. Although this section is intended to be self-contained, we will not derive many equations and the reader is referred to any continuum mechanics text (e.g., Malvern, 1969) for further details.

Deformations in three-dimensional materials are termed strain; internal forces between different parts of the medium are called stress. Stress and strain do not exist independently in materials; they are linked through the constitutive relationships that describe the nature of elastic solids.

2.1 The Stress Tensor

Consider an infinitesimal plane of arbitrary orientation within a homogenous elastic medium in static equilibrium. The orientation of the plane may be specified by its unit normal vector, $\hat{\mathbf{n}}$. The force per unit area exerted by the side in the direction of $\hat{\mathbf{n}}$ across this plane is termed the *traction* and is represented by the vector $\mathbf{t}(\hat{\mathbf{n}}) = (t_x, t_y, t_z)$. There is an equal and opposite force exerted by the side opposing $\hat{\mathbf{n}}$, such that $\mathbf{t}(-\hat{\mathbf{n}}) = -\mathbf{t}(\hat{\mathbf{n}})$. The part of $\mathbf{t}$ that is normal to the plane is termed the *normal stress*; that parallel to it is called the *shear stress*. In the case of a fluid, there are no shear stresses and $\mathbf{t} = -P\hat{\mathbf{n}}$, where P is the pressure.



The stress tensor, $\boldsymbol{\tau}$, in a Cartesian coordinate system (Fig. 2.1) may be defined† by the tractions across the yz, xz, and xy planes:

$$\boldsymbol{\tau} = \begin{bmatrix} t_x(\hat{\mathbf{x}}) & t_x(\hat{\mathbf{y}}) & t_x(\hat{\mathbf{z}}) \\ t_y(\hat{\mathbf{x}}) & t_y(\hat{\mathbf{y}}) & t_y(\hat{\mathbf{z}}) \\ t_z(\hat{\mathbf{x}}) & t_z(\hat{\mathbf{y}}) & t_z(\hat{\mathbf{z}}) \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}. \tag{2.1}$$

† Often the stress tensor is defined as the transpose of (2.1) so that the first subscript of $\boldsymbol{\tau}$ represents the surface normal direction. In practice, it makes no difference as $\boldsymbol{\tau}$ is symmetric.

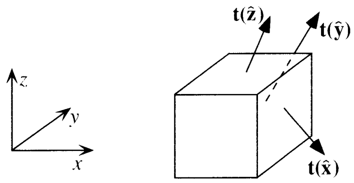
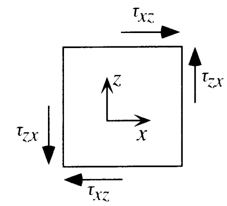


Fig. 2.1. The traction vectors $\mathbf{t}(\hat{\mathbf{x}})$, $\mathbf{t}(\hat{\mathbf{y}})$, and $\mathbf{t}(\hat{\mathbf{z}})$ describe the forces on the faces of an infinitesimal cube in a Cartesian coordinate system.

Because the solid is in static equilibrium, there can be no net rotation from the shear stresses. For example, consider the shear stresses in the xz plane. To balance the torques, $\tau_{xz} = \tau_{zx}$. Similarly, $\tau_{xy} = \tau_{yx}$ and $\tau_{yz} = \tau_{zy}$, and the stress tensor $\boldsymbol{\tau}$ is symmetric, that is,



$$\boldsymbol{\tau} = \boldsymbol{\tau}^T = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}. \tag{2.2}$$

The stress tensor $\boldsymbol{\tau}$ contains only six independent elements, and these are sufficient to completely describe the state of stress at a given point in the medium.

The traction across any arbitrary plane of orientation defined by $\hat{\mathbf{n}}$ may be obtained by multiplying the stress tensor by $\hat{\mathbf{n}}$, that is,

$$\mathbf{t}(\hat{\mathbf{n}}) = \boldsymbol{\tau}\hat{\mathbf{n}} = \begin{bmatrix} t_x(\hat{\mathbf{n}}) \\ t_y(\hat{\mathbf{n}}) \\ t_z(\hat{\mathbf{n}}) \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \tau_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \begin{bmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{bmatrix}. \tag{2.3}$$

This can be shown by summing the forces on the surfaces of a tetrahedron (the *Cauchy tetrahedron*) bounded by the plane normal to $\hat{\mathbf{n}}$ and the xy, xz, and yz planes.

The stress tensor is simply the linear operator that produces the traction vector $\mathbf{t}$ from the normal vector $\hat{\mathbf{n}}$, and, in this sense, the stress tensor exists independent of any particular coordinate system. In seismology we almost always write the stress tensor as a 3×3 matrix in a Cartesian geometry. Note that the symmetry requirement reduces the number of independent parameters in the stress tensor to six from the nine that are present in the most general form of a second-order tensor (scalars are considered zeroth-order tensors, vectors are first order, etc.).

The stress tensor will normally vary with position in a material; it is a measure of the forces acting on infinitesimal planes at each point in the solid. Stress provides a measure only of the forces exerted across these planes and has units of force per unit area. However, other forces may be present (e.g., gravity); these are termed *body forces* and have units of force per unit volume or mass.

For any stress tensor, it is always possible to find a direction $\hat{\mathbf{n}}$ such that there are no shear stresses across the plane normal to $\hat{\mathbf{n}}$, that is, $\mathbf{t}(\hat{\mathbf{n}})$ is in the $\hat{\mathbf{n}}$ direction.

In this case

$$\begin{aligned} \mathbf{t}(\hat{\mathbf{n}}) &= \lambda\hat{\mathbf{n}} = \boldsymbol{\tau}\hat{\mathbf{n}}, \\ \boldsymbol{\tau}\hat{\mathbf{n}} - \lambda\hat{\mathbf{n}} &= 0, \\ (\boldsymbol{\tau} - \mathbf{I}\lambda)\hat{\mathbf{n}} &= 0, \end{aligned} \tag{2.4}$$

where $\mathbf{I}$ is the identity matrix and λ is a scalar. This is an eigenvalue problem that has a nontrivial solution only when

$$\det[\boldsymbol{\tau} - \mathbf{I}\lambda] = 0. \tag{2.5}$$

This is a cubic equation with three solutions, the eigenvalues λ_1, λ_2, and λ_3 (do not confuse these with the Lamé parameter λ that we will discuss later). Since $\boldsymbol{\tau}$ is symmetric and real, the eigenvalues are real. Corresponding to the eigenvalues are the eigenvectors $\hat{\mathbf{n}}^{(1)}$, $\hat{\mathbf{n}}^{(2)}$, and $\hat{\mathbf{n}}^{(3)}$. The eigenvectors are orthogonal and define the *principal axes* of stress. The planes perpendicular to these axes are termed the *principal planes*. We can rotate $\boldsymbol{\tau}$ into the $\hat{\mathbf{n}}^{(1)}$, $\hat{\mathbf{n}}^{(2)}$, $\hat{\mathbf{n}}^{(3)}$ coordinate system by applying a similarity transformation:

$$\boldsymbol{\tau}^R = \mathbf{N}^T\boldsymbol{\tau}\mathbf{N} = \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_3 \end{bmatrix}, \tag{2.6}$$

where $\boldsymbol{\tau}^R$ is the rotated stress tensor and τ_1, τ_2, and τ_3 are the *principal stresses* (identical to the eigenvalues λ_1, λ_2, and λ_3). Here $\mathbf{N}$ is the matrix of eigenvectors

$$\mathbf{N} = \begin{bmatrix} n_x^{(1)} & n_x^{(2)} & n_x^{(3)} \\ n_y^{(1)} & n_y^{(2)} & n_y^{(3)} \\ n_z^{(1)} & n_z^{(2)} & n_z^{(3)} \end{bmatrix}, \tag{2.7}$$

with $\mathbf{N}^T = \mathbf{N}^{-1}$ for orthogonal eigenvectors normalized to unit length.

If $\tau_1 = \tau_2 = \tau_3$, then the stress field is called *hydrostatic* and there are no planes of any orientation in which shear stress exists. In a fluid the stress tensor can be written

$$\boldsymbol{\tau} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix}, \tag{2.8}$$

where P is the pressure.

2.1.1 Values for Stress

Stress has units of force per unit area. In SI units

$$1 \text{ pascal (Pa)} = 1 \text{ N m}^{-2}.$$

Table 2.1. Pressure Versus Depth Inside Earth

Depth (km)	Region	Pressure (GPa)
0–24	Crust	0–0.6
24–400	Upper Mantle	0.6–13.4
400–670	Transition Zone	13.4–23.8
670–2891	Lower Mantle	23.8–135.8
2891–5150	Outer Core	135.8–328.9
5150–6371	Inner Core	328.9–363.9

Recall that 1 newton (N) = 1 kg m s^{-2} = 10^5 dyne. Another commonly used unit for stress is the *bar*:

$$
\begin{aligned}
1\ \text{bar} &= 10^5\ \text{Pa}, \\
1\ \text{kbar} &= 10^8\ \text{Pa} = 100\ \text{MPa}, \\
1\ \text{Mbar} &= 10^{11}\ \text{Pa} = 100\ \text{GPa}.
\end{aligned}
$$

Pressure increases rapidly with depth in Earth, as shown in Table 2.1 using values taken from the reference model PREM (Dziewonski and Anderson, 1981). Pressures reach 13.4 GPa at 400 km depth, 136 GPa at the core–mantle boundary, and 329 GPa at the inner-core boundary. In contrast, the pressure at the center of the Moon is only about 4.8 GPa, a value reached in Earth at 150 km depth (Latham et al., 1969). This is a result of the much smaller mass of the Moon.

These are the hydrostatic pressures inside Earth; shear stresses at depth are much smaller in magnitude and include stresses associated with mantle convection and the dynamic stresses caused by seismic wave propagation. Static shear stresses can be maintained in the upper, brittle part of the crust. Measuring shear stress in the crust is a topic of current research and the magnitude of the stress is a subject of some controversy. Crustal shear stress is probably between about 100 and 1,000 bars (10 to 100 MPa), with a tendency for lower stresses to occur close to active faults (which act to relieve the stress).

2.2 The Strain Tensor

Now let us consider how to describe changes in the positions of points within a continuum. The location of every point relative to its position at a reference time t_0 can be expressed as a vector field, that is, the displacement field $\mathbf{u}$ is given by

$$\mathbf{u}(\mathbf{r}_0) = \mathbf{r} - \mathbf{r}_0, \tag{2.9}$$

where $\mathbf{r}$ is the current position of the point and $\mathbf{r}_0$ is the reference location of the point. The displacement field is an important concept and we will refer to it often in this book. It is an absolute measure of position changes. In contrast, *strain* is a local measure of relative changes in the displacement field, that is, the spatial gradients in the displacement field. Strain is related to the deformation, or change in shape, of a material rather than any absolute change in position. For example, *extensional strain* is defined as the change in length with respect to length. If a 100 m long string is fixed at one end and uniformly stretched to a length of 101 m, then the displacement field varies from 0 to 1 m along the string, whereas the strain field is constant at 0.01 (1%) everywhere in the string.

Consider the displacement $\mathbf{u} = (u_x, u_y, u_z)$ at position $\mathbf{x}$, a small distance away from a reference position $\mathbf{x}_0$. We can expand $\mathbf{u}$ in a Taylor series to obtain

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \mathbf{u}(\mathbf{x}_0) + \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \mathbf{u}(\mathbf{x}_0) + \mathbf{J}\mathbf{d}, \tag{2.10}$$

where $\mathbf{d} = \mathbf{x} - \mathbf{x}_0$. We have ignored higher order terms in the expansion by assuming that the partials, $\partial u_x/\partial x$, $\partial u_y/\partial x$, etc., are small enough that their products can be ignored (the basis for *infinitesimal strain theory*). Seismology is fortunate that actual Earth strains are almost always small enough that this approximation is valid. We can separate out rigid rotations by dividing $\mathbf{J}$ into symmetric and antisymmetric parts:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} = \mathbf{e} + \boldsymbol{\Omega}, \tag{2.11}$$

where the *strain tensor*, $\mathbf{e}$, is symmetric ($e_{ij} = e_{ji}$) and is given by

$$\mathbf{e} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}, \tag{2.12}$$

and the *rotation tensor*, $\boldsymbol{\Omega}$, is antisymmetric ($\Omega_{ij} = -\Omega_{ji}$) and is given by

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ -\frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) \\ -\frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) & -\frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) & 0 \end{bmatrix}. \tag{2.13}$$

The reader should verify that $\mathbf{e} + \boldsymbol{\Omega} = \mathbf{J}$.

The effect of $\mathbf{e}$ and $\boldsymbol{\Omega}$ may be illustrated by considering what happens to an infinitesimal cube (Fig. 2.2). The off-diagonal elements of $\mathbf{e}$ cause shear strain; for example, in two-dimensions, if $\boldsymbol{\Omega} = \mathbf{0}$ and there is no volume change, then $\partial u_x/\partial x = \partial u_z/\partial z = 0$, $\partial u_x/\partial z = \partial u_z/\partial x$, and

$$\mathbf{J} = \mathbf{e} = \begin{bmatrix} 0 & \theta \\ \theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_z}{\partial x} & 0 \end{bmatrix}, \tag{2.14}$$

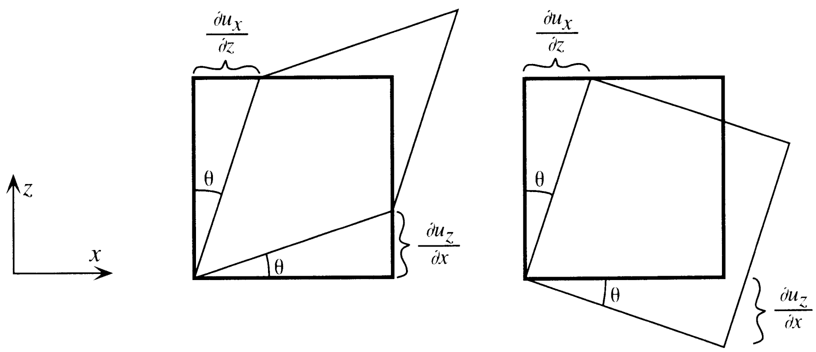


Fig. 2.2. The different effects of the strain tensor **e** and the rotation tensor **Ω** are illustrated by the deformation of a square in the $x-z$ plane. The off-diagonal components of **e** cause shear deformation (left square), whereas **Ω** causes rigid rotation (right square). The deformations shown here are highly exaggerated compared to those for which infinitesimal strain theory is valid.

where θ is the angle (in radians, not degrees!) through which each side rotates. Note that the total change in angle between the sides is 2θ. In contrast, the **Ω** matrix causes rigid rotation, for example, if $\mathbf{e} = \mathbf{0}$, then $\partial u_x/\partial z = -\partial u_z/\partial x$ and

$$\mathbf{J} = \mathbf{\Omega} = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_z}{\partial x} & 0 \end{bmatrix}. \tag{2.15}$$

In both of these cases there is no volume change in the material. The relative volume increase, or *dilatation*, $\Delta = (V - V_0)/V_0$, is given by the sum of the extensions in the x, y, and z directions:

$$\Delta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \mathrm{tr}[\mathbf{e}] = \nabla \cdot \mathbf{u}, \tag{2.16}$$

where $\mathrm{tr}[\mathbf{e}] = e_{11} + e_{22} + e_{33}$, the *trace* of **e**. Note that the dilatation is given by the divergence of the displacement field.

What about the curl of the displacement field? Recall the definition of the curl of a vector field:

$$\nabla \times \mathbf{u} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\hat{\mathbf{z}}. \tag{2.17}$$

A comparison of this equation with (2.13) shows that $\nabla \times \mathbf{u}$ is nonzero only if **Ω** is nonzero and the displacement field contains some rigid rotation.

The strain tensor, like the stress tensor, is symmetric and contains six independent parameters. The *principal axes* of strain may be found by computing the directions $\hat{\mathbf{n}}$ for which the displacements are in the same direction, that is,

$$\mathbf{u} = \lambda\hat{\mathbf{n}} = \mathbf{e}\hat{\mathbf{n}}. \tag{2.18}$$

This is analogous to the case of the stress tensor discussed in the previous section. The three eigenvalues are the *principal strains*, e_1, e_2, and e_3, while the eigenvectors define the principal axes. Note that, except in the case $e_1 = e_2 = e_3$

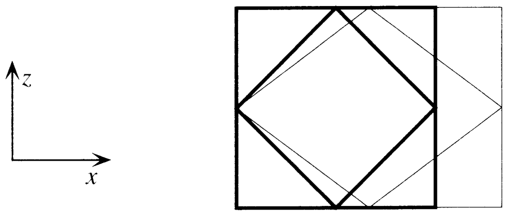


Fig. 2.3. Simple extensional strain in the x direction results in shear strain; internal angles are not preserved.

(*hydrostatic strain*), there is always some shear strain present. For example, consider a two-dimensional square with extension only in the x direction (Fig. 2.3), so that $\mathbf{e}$ is given by

$$\mathbf{e} = \begin{bmatrix} e_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.19}$$

Angles between lines parallel to the coordinate axes do not change, but lines at intermediate angles are seen to rotate. The angle changes associated with shearing would become obvious if the coordinate system were rotated by 45 degrees (in which case $\mathbf{e}$ would have off-diagonal terms).

In subsequent sections, we will find it helpful to express the strain tensor using index notation. Equation (2.12) can be rewritten as

$$e_{ij} = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i), \tag{2.20}$$

where i and j are assumed to range from 1 to 3 (for the x, y, and z directions) and we are using the notation $\partial_x u_y = \partial u_y / \partial x$.

2.2.1 Values for Strain

Strain is dimensionless since it represents a change in length divided by length. Dynamic strains associated with the passage of seismic waves in the far field are typically less than 10^{-6}.

2.3 The Linear Stress–Strain Relationship

Stress and strain are linked in elastic media by a stress–strain or *constitutive* relationship. The most general linear relationship between the stress and strain tensors can be written

$$\tau_{ij} = c_{ijkl} e_{kl} \equiv \sum_{k=1,3} \sum_{l=1,3} c_{ijkl} e_{kl}, \tag{2.21}$$

where c_{ijkl} is termed the *elastic tensor*. Here we begin using the *summation convention* in our index notation. Any repeated index in a product indicates that the sum is to be taken as the index varies from 1 to 3. Equation (2.21) assumes perfect elasticity; there is no energy loss or attenuation as the material deforms in response to the applied stress (sometimes these effects are modeled by permitting c_{ijkl} to be complex). We will not consider anelastic behavior and attenuation until Chapter 6.

The elastic tensor, c_{ijkl}, is a fourth-order tensor with 81 (3^4) components. However, because of the symmetry of the stress and strain tensors and thermodynamic considerations, only 21 of these components are independent. These 21 components are necessary to specify the stress–strain relationship for the most general form of elastic solid. The properties of such a solid may vary with direction; if they do, the material is termed *anisotropic*. In contrast, the properties of an *isotropic* solid are the same in all directions. Isotropy has proven to be a reasonable first-order approximation for much of the Earth's interior, but in some regions anisotropy has been observed and this is an important area of current research (see Section 11.3 for more about anisotropy).

If we assume isotropy (c_{ijkl} is invariant with respect to rotation), it can be shown that the number of independent parameters is reduced to two:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}), \tag{2.22}$$

where λ and μ are called the *Lamé parameters* of the material and δ_{ij} is the *Kronecker delta* ($\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$). As we shall see, the Lamé parameters, together with the density, will eventually determine the seismic velocities of the material. The stress–strain equation (2.21) for an isotropic solid is

$$\begin{aligned} \tau_{ij} &= [\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})]e_{kl} \\ &= \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}, \end{aligned} \tag{2.23}$$

where we have used $e_{ij} = e_{ji}$ to combine the μ terms. Note that $e_{kk} = \mathrm{tr}[\mathbf{e}]$, the sum of the diagonal elements of $\mathbf{e}$. Using this equation, we can directly write the components of the stress tensor in terms of the strains:

$$\boldsymbol{\tau} = \begin{bmatrix} \lambda\,\mathrm{tr}[\mathbf{e}] + 2\mu e_{11} & 2\mu e_{12} & 2\mu e_{13} \\ 2\mu e_{21} & \lambda\,\mathrm{tr}[\mathbf{e}] + 2\mu e_{22} & 2\mu e_{23} \\ 2\mu e_{31} & 2\mu e_{32} & \lambda\,\mathrm{tr}[\mathbf{e}] + 2\mu e_{33} \end{bmatrix}. \tag{2.24}$$

The two Lamé parameters completely describe the linear stress–strain relation within an isotropic solid. μ is termed the *shear modulus* and is a measure of the resistance of the material to shearing. Its value is given by half of the ratio between the applied shear stress and the resulting shear strain, that is, $\mu = \tau_{xy}/2e_{xy}$. The other Lamé parameter, λ, does not have a simple physical explanation. Other commonly used elastic constants for isotropic solids include:

Young's modulus E: The ratio of extensional stress to the resulting extensional strain for a cylinder being pulled on both ends. It can be shown that

$$E = \frac{(3\lambda + 2\mu)\mu}{(\lambda + \mu)}. \tag{2.25}$$

Bulk modulus κ: The ratio of hydrostatic pressure to the resulting volume change, a measure of the incompressibility of the material. It can be expressed as

$$\kappa = \lambda + \tfrac{2}{3}\mu. \tag{2.26}$$

Poisson's ratio σ: The ratio of the lateral contraction of a cylinder (being pulled on its ends) to its longitudinal extension. It can be expressed as

$$\sigma = \frac{\lambda}{2(\lambda + \mu)}. \tag{2.27}$$

In seismology, we are mostly concerned with the compressional (P) and shear (S) velocities. As we will show later, these can be computed from the elastic constants and the density, ρ:

P velocity, α, can be expressed as

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{2.28}$$

S velocity, β, can be expressed as

$$\beta = \sqrt{\frac{\mu}{\rho}}. \tag{2.29}$$

Poisson's ratio σ is often used as a measure of the relative size of the P and S velocities; it can be shown that

$$\sigma = \frac{\alpha^2 - 2\beta^2}{2(\alpha^2 - \beta^2)}. \tag{2.30}$$

Note that σ is dimensionless and varies between 0 and 0.5 with the upper limit representing a fluid ($\mu = 0$). For a *Poisson solid*, $\lambda = \mu$, $\sigma = 0.25$, and $\alpha/\beta = \sqrt{3}$. Most crustal rocks have Poisson's ratios between 0.25 and 0.30.

2.3.1 Units for Elastic Moduli

The Lamé parameters, Young's modulus, and the bulk modulus all have the same units as stress (i.e., pascals). Recall that

$$1 \text{ Pa} = 1 \text{ N m}^{-2} = 1 \text{ kg m}^{-1}\text{s}^{-2}.$$

Note that when this is divided by density (kg m^{-3}) the result is units of velocity squared (appropriate for Equations 2.28 and 2.29).

EXERCISES

2.1 Using Equations (2.4), (2.18), and (2.24), show that the principal stress axes always coincide with the principal strain axes for isotropic media.

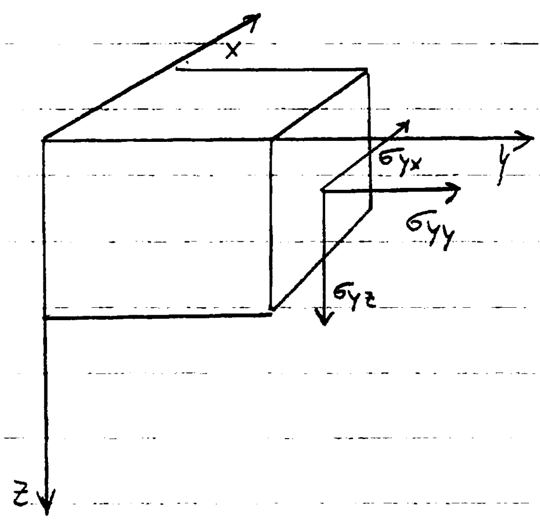
BRIEF REVIEW OF ELASTICITY (Shearer, Ch 2; T&S, Ch 3)

This is a very brief review of elasticity theory. I assume that you have already taken continuum mechanics.

There is a change in sign between the way seismologists and engineers define stress and the way geologists define stress. T&S use the geological notation.

compression positive - geology
compression negative - other fields

<u>stress</u> - force / area ($N\ m^{-2} = Pa$)



full stress tensor is symmetric so there are only 6 independent components

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = \sigma_{ij}$$

strain - $\dfrac{\text{change in length}}{\text{length}}$ (dimensionless)

$\vec{u} = (u_x, u_y, u_z) = u_i$ displacement

$\nabla \vec{u} = u_{i,j}$ $\qquad u_{x,y} = \dfrac{\partial u_x}{\partial y}$

$$\nabla \vec{u} = \underset{\text{strain}}{\underline{\underline{e}}} + \underset{\text{rotation}}{\underline{\underline{\Omega}}}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

stress vs strain

$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij}$ assumes isotropic and symmetric

shear modulus μ or G in T&S

$\sigma_{xy} = 2\mu \epsilon_{xy} = \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{y}\right)$

relates shear stress to shear strain

principal stress - one can always find a rotation of the co-ordinate system such that all of the shear stresses are zero.

$$\vec{\sigma}_p = R^T \underline{\underline{\sigma}} R = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \sigma_3$

pressure $p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$

$\tau = \frac{1}{2}(\sigma_1 - \sigma_3)$ maximum shear stress principal stress and strain

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix} = \begin{pmatrix}\lambda+2\mu & \lambda & \lambda\\ \lambda & \lambda+2\mu & \lambda\\ \lambda & \lambda & \lambda+2\mu\end{pmatrix}\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\end{pmatrix}$$

uniaxial stress $\quad \sigma_2 = \sigma_3 = 0 \qquad \epsilon_2 = \epsilon_3$

$$0 = \lambda\epsilon_1 + (\lambda+2\mu)\epsilon_2 + \lambda\epsilon_2$$

$$\epsilon_2 = \frac{-\lambda}{2(\lambda+\mu)}\epsilon_1 \qquad \epsilon_2 = -\nu\epsilon_1 \quad \text{Poisson's ratio}$$

$$\sigma_1 = (\lambda+2\mu)\epsilon_1 + \frac{--\lambda^2}{\lambda+\mu}\epsilon_1$$

$$\sigma_1 = \frac{(\lambda+2\mu)(\lambda+\mu)-\lambda^2}{(\lambda+\mu)}\epsilon_1 = \frac{\cancel{\lambda^2}+3\lambda\mu+2\mu^2-\cancel{\lambda^2}}{(\lambda+\mu)}$$

$$\sigma_1 = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}\epsilon_1 \qquad \sigma_1 = E\,\epsilon_1 \quad \text{Young's modulus}$$

uniform compression $\quad \sigma_1 = \sigma_2 = \sigma_3 \qquad \Delta P = \sigma$

$$\Delta V = \epsilon_1 + \epsilon_2 + \epsilon_3$$

$$\frac{\Delta P}{\Delta V} = K \quad \text{bulk modulus}$$

add 3 equations

$$3\sigma_1 = (3\lambda + 2\mu)\, 3\epsilon_1$$

$$\Delta P = \left(\lambda + \tfrac{2}{3}\mu\right) \Delta V$$

$$\Delta P = K \Delta V$$ K - bulk modulus

save

Assume x y z co ordinate system is aligned with principal axes

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix}$$

<u>plane stress</u> $\sigma_{zz} = 0$

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu \sigma_{yy})$$

$$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu \sigma_{xx})$$

$$\epsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy})$$