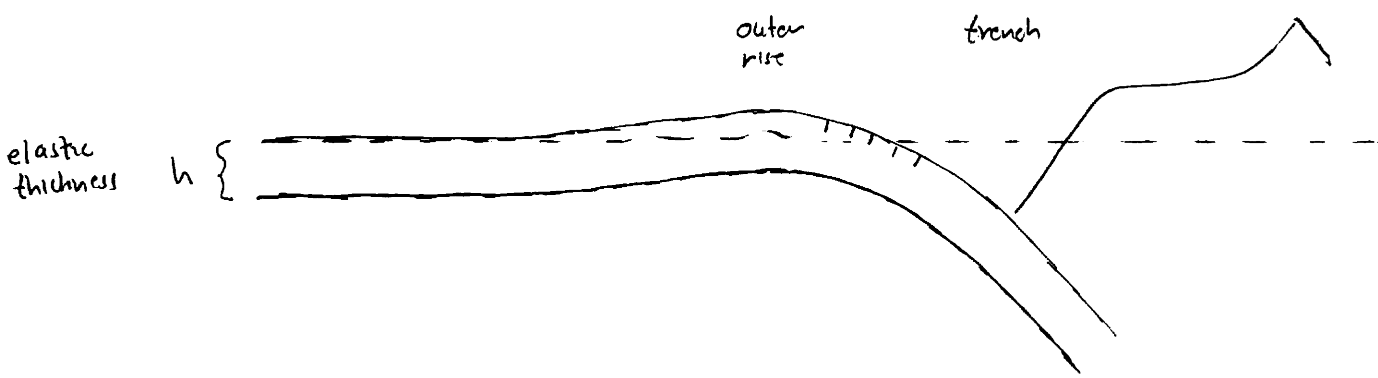
FLEXURE OF THE LITHOSPHERE (T&S, CH3)

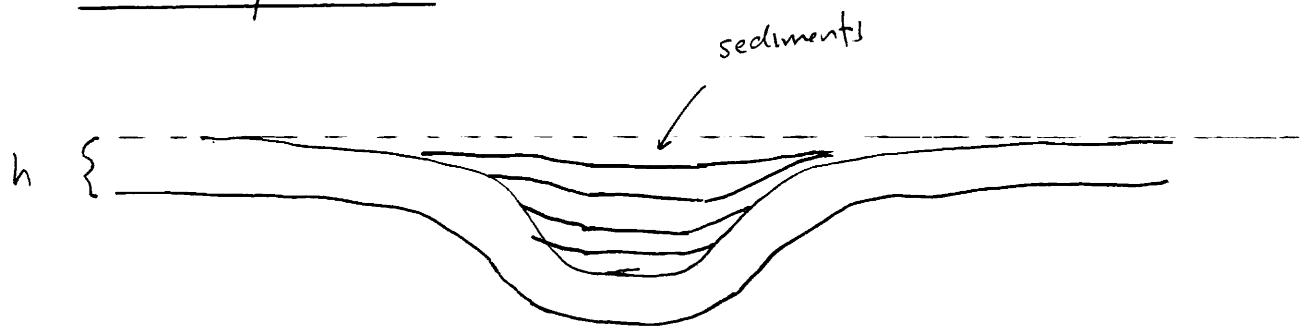
(Watts, Isostasy and Flexure of the Lithosphere, Cambridge 2001)

How does the outer shell of the earth respond to vertical and end loads?
What is the long-term thickness of the lithosphere?
What is the largest load that can be supported by the lithosphere?

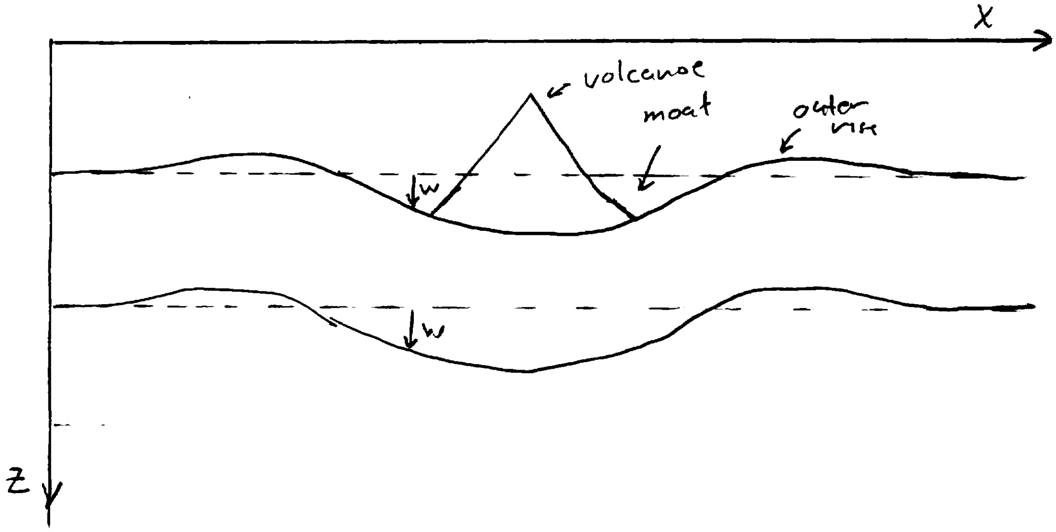
ocean trench



sedimentary basin



seamount load



w – deflection of the plate

D – flexural rigidity

$$D = \frac{E h^3}{12(1-\nu^2)}$$

$$\frac{\partial}{\partial x^2} D \frac{\partial^2 w}{\partial x^2} + P \frac{\partial^2 w}{\partial x^2} + g \Delta\rho\, w(x) = q(x)$$

strength of plate

end load

restoring force

applied load volcanoe

Airy isostasy

$$M = D \frac{\partial^2 w}{\partial x^2}$$

bending moment

plate curvature

last class

principal stress and strain $\quad \lambda, \mu$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} \lambda+\mu & \lambda & \lambda \\ \lambda & \lambda+\mu & \lambda \\ \lambda & \lambda & \lambda+\mu \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}$$

invert $\Downarrow$

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \qquad \nu = \frac{\lambda}{2(\lambda+\mu)} \qquad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

align principal stress/strain axes with x y z

save

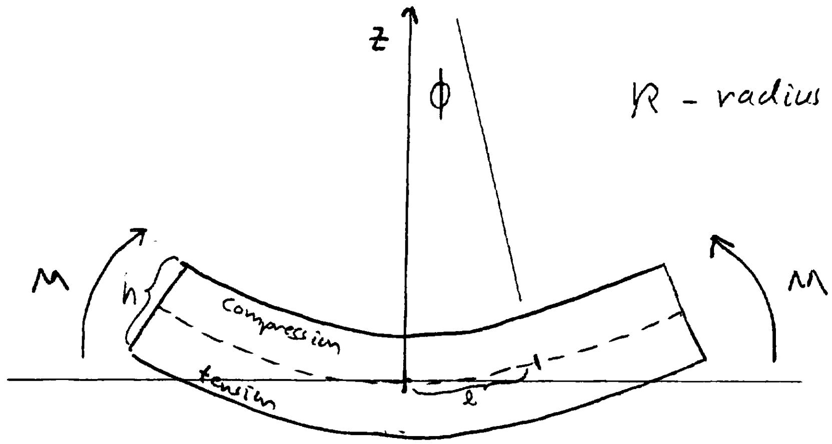
plane stress $\quad \sigma_{zz} = 0$

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu \sigma_{yy})$$

$$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu \sigma_{xx})$$

$$\epsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

moment vs. curvature (T&S p 113-115)



assume 2-dimensional bending of plate

assume thin plate so $\sigma_{zz} = 0$ - plane stress approximation

assume small displacement

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\, z \, dz$$

moment per unit length in y-direction

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu \sigma_{yy}\right)$$

$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu \sigma_{xx}\right)$$

$\epsilon_{yy} = 0 \Rightarrow \sigma_{yy} = \nu\, \sigma_{xx}$

$$\epsilon_{xx} = \frac{1-\nu^2}{E}\, \sigma_{xx}$$

$$M = \frac{E}{(1-\nu^2)} \int_{-\frac{h}{2}}^{\frac{h}{2}} \epsilon_{xx}\, z \, dz$$

$$\epsilon_{xx} = \frac{-\Delta \ell}{\ell}$$

$$\ell(0) = R\,\phi$$
$$\ell(z) = (R-z)\,\phi$$
$$\Delta\ell(z) = -z\,\phi$$

$$\epsilon_{xx} = \frac{z}{R} = -z\,\frac{d^2 w}{dx^2}$$

$$M = \frac{-E}{(1-\nu^2)}\,\frac{d^2 w}{dx^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\,dz \qquad \left.\frac{z^3}{3}\right|_{-\frac{h}{2}}^{\frac{h}{2}} = \frac{h^3}{12}$$

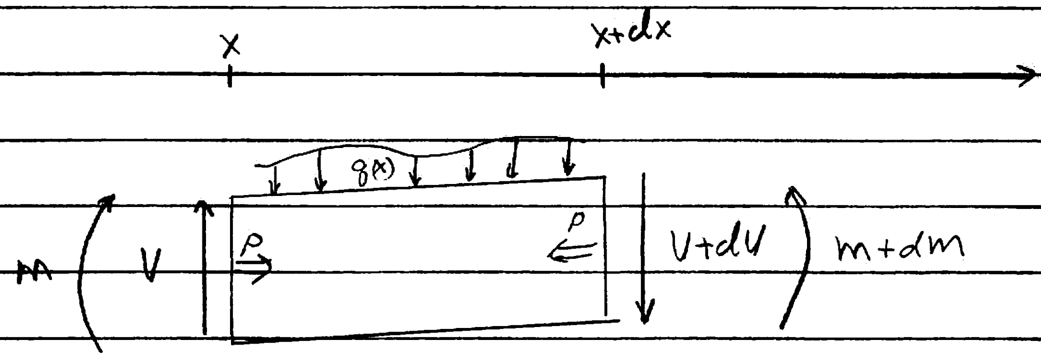
$$M = \underbrace{\frac{-E\,h^3}{12(1-\nu^2)}}_{=D}\;\frac{d^2 w}{dx^2}$$

moment (force) — flexural rigidity (force x length) — curvature length^{-1}

All of the above problems can be approximated by flexure of a thin elastic plate in response to various types of loads.

2-D BENDING OR FLEXURE OF PLATES (T & S p. 113-115)

Deflection of a plate $w(x)$ can be determined by requiring equilibrium under all forces and torques.



$w(x)$ - deflection of plate (length)
$q(x)$ - downward (force/area)
V - shear (force/length)
m - bending moment/length (force length/length = force)

vertical force balance:

$$q(x)\,dx + dV = 0 \qquad \text{or} \qquad \frac{dV}{dx} = -q$$

moment balance

$$dm - P\,dw = V\,dx \qquad \text{or} \qquad \frac{dm}{dx} = V + P\frac{dw}{dx}$$

take derivative w.r.t. x

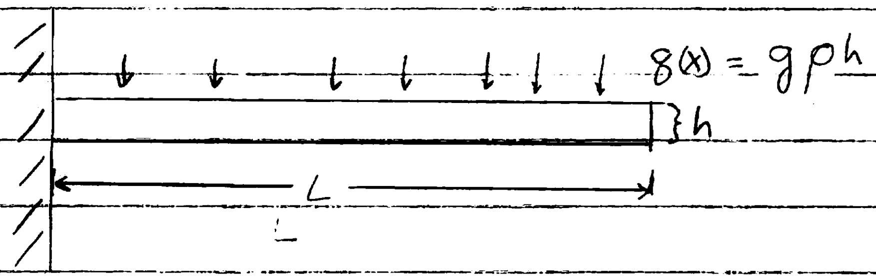
$$\frac{d^2 m}{dx^2} = \frac{dV}{dx} + P\frac{d^2 w}{dx^2} \qquad \text{but } \frac{dV}{dx} = -q$$

$$\frac{d^2 m}{dx^2} = -q(x) + P\frac{d^2 w}{dx^2}$$

but $\quad m = -D\dfrac{d^2 w}{dx^2}$

$$\boxed{D\frac{d^4 w}{dx^4} + P\frac{d^2 w}{dx^2} = q(x)}$$

Example - measuring flexural rigidity of a ruler



$$D\frac{d^4 w}{dx^4} = g\rho h$$

$w(0) = 0$

$\dfrac{dw}{dx}(0) = 0$

$\dfrac{d^2 w}{dx^2}(L) = 0$ no moment

$\dfrac{d^3 w}{dx^3}(L) = 0$ no load at $x = L$

$$\frac{d^3w}{dx^3} = \frac{\rho g h}{D}(x + c_1) \qquad c_1 = -L$$

$$\frac{d^2w}{dx^2} = \frac{\rho g h}{D}\left(\frac{x^2}{2} - Lx + c_2\right) \qquad c_2 = \frac{L^2}{2}$$

$$\frac{dw}{dx} = \frac{\rho g h}{D}\left(\frac{x^3}{6} - \frac{Lx^2}{2} + \frac{L^2x}{2} + c_3\right) \qquad c_3 = 0$$

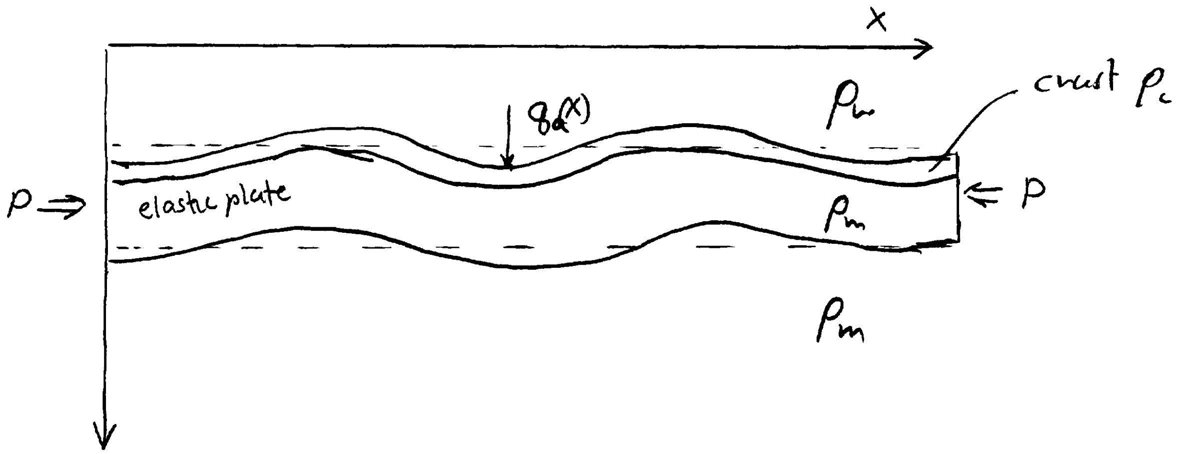
$$w = \frac{\rho g h}{D}\left(\frac{x^4}{24} - \frac{Lx^3}{6} + \frac{L^2x^2}{4}\right) \qquad c_4 = 0$$

extend ruler over edge of table and measure $w(L)$

$$w(L) = \frac{\rho g h}{D}\frac{L^4}{8} \qquad \text{but } D = \frac{E h^3}{12(1-\nu^2)}$$

$$\frac{E\, \cancel{h^3}^{\,2}}{12(1-\nu^2)} = \frac{\rho g \cancel{h}}{w(L)}\frac{L^4}{8} \qquad \boxed{E = \frac{3(1-\nu^2)\rho g L^4}{w(L)\, h^2}}$$

stability of an elastic plate under an end load



$$D \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} + g(\rho_m - \rho_w) w = q_a(x)$$

take fourier transform

$$(2\pi k)^4 D\, W(k) - (2\pi k)^2 P\, W(k) + (\rho_m - \rho_w) g\, W(k) = 0$$

$$D(2\pi k)^4 - P(2\pi k)^2 + (\rho_m - \rho_w) g = 0 \qquad ay^2 + by + c = 0$$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$(2\pi k)^2 = \frac{P \pm \sqrt{P^2 - 4D(\rho_m - \rho_w) g}}{2D}$$

$$P > \left[4Dg(\rho_m - \rho_w)\right]^{\frac{1}{2}}$$

wavenumber must be real so $P > \left[4Dg(\rho_m-\rho_w)\right]^{\frac{1}{2}}$

critical wavenumber $P_c = \left[4Dg(\rho_m-\rho_w)\right]^{\frac{1}{2}}$ $\quad \sigma_c = \dfrac{P_c}{h}$

critical wavelength $(2\pi k_c)^2 = \dfrac{\left[4Dg(\rho_m-\rho_w)\right]^{\frac{1}{2}}}{2D}$

$$\boxed{\lambda_c = \frac{1}{k_c} = 2\pi\left(\frac{D}{g(\rho_m-\rho_w)}\right)^{\frac{1}{4}}}$$

later we'll see this is the flexural wavelength

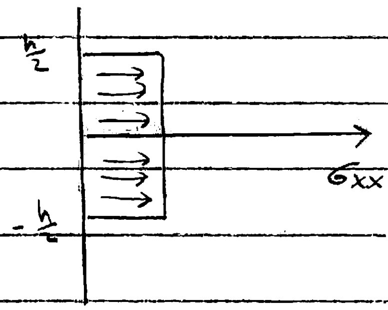
<u>Example</u> $h = 50\,km$ $\quad E = 10^{11}\,Pa$ $\quad \nu = .25$

$\rho_m = 3300$ $\quad \rho_w = 1025$

$\sigma = P_c/h = 6.4\,GPa$ impossible!

$\lambda_c = 513\,km$

<u>FINITE STRENGTH PLATE</u>



$\sigma_{xx} = \dfrac{P}{h}$

<u>rock - brittle</u> <u>steel - plastic</u>