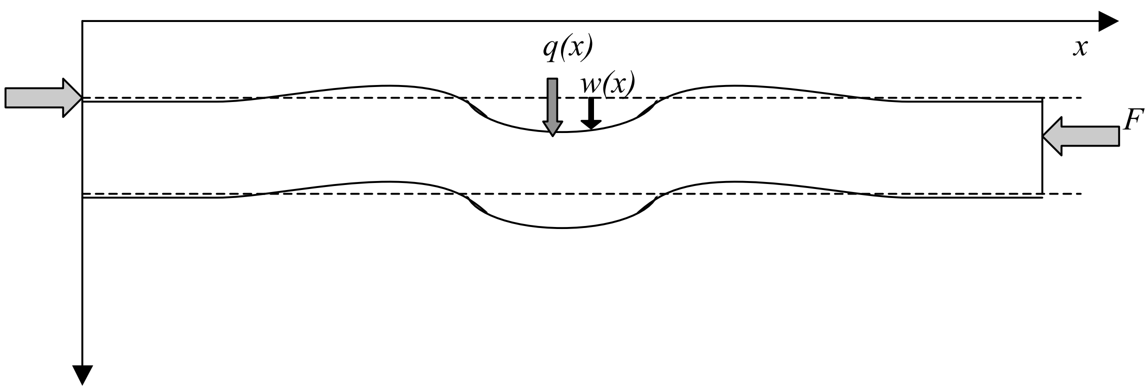
FLEXURE OF THE LITHOSPHERE

(Reference: Turcotte and Schubert, Chapter 3)

These lecture notes are basically a supplement to Turcotte and Schubert, Chapter 3. The results of the first derivation are the same as equation 3-130 in T&S but rather than guessing the general solution, the solution is developed using fourier transforms. The approach is similar to the solutions of the marine magnetic anomaly problem, the lithospheric heat conduction problem, the strike-slip fault flexure problem and the flat-earth gravity problem. In all these cases, we use the Cauchy integral theorem to perform the inverse fourier transform. Later we'll combine this flexure solution with the gravity solution to develop the gravity-to-topography transfer function. Moreover, one can take this approach further to develop a Green's function relating temperature, heat flow, topography and gravity to a point heat source (e.g., Sandwell, Thermal Isostasy: response of a Moving Lithosphere to a Distributed Heat Source*, J. Geophys. Res.,* .v 87, p. 1001-1014, 1982). In addition to the constant flexural rigidity solution found in the literature, we develop an iterative solution to flexure with spatially-variable rigidity.

Before going over these notes, please re-read section 3-9 in Turcotte and Schubert on the development of moment versus curvature for a thin elastic plate.



The loading problem is illustrated above. We start with a simple line source, but the solution method also applies to a point source. Of course, the point source Green's function can be convolved with an arbitrary load distribution to make the solution completely general; we'll do this later. The vertical force balance for flexure of a thin elastic plate floating on the mantle is described by the following differential

$$\underbrace{\frac{d^2}{dx^2}\left(D(x)\frac{d^2w}{dx^2}\right)}_{\text{flexural resistance}} + \underbrace{F\frac{d^2w}{dx^2}}_{\text{end load}} + \underbrace{\Delta\rho g w}_{\text{restoring force}} = \underbrace{q(x)}_{\text{vertical load}} \qquad (1)$$

Parameter	Definition	Value/Unit
$w(x)$	deflection of plate (positive down)	m
$D(x)$	flexural rigidity	N m
h	elastic plate thickness	m
F	end load	N m^{-1}
q	vertical load	N m^{-2}
$\Delta\rho$	density contrast (ρ_m - ρ_w)	
g	acceleration of gravity	9.82 m s^{-2}
E	Young's modulus	6.5 x 10^{10} Pa
ν	Poisson's ratio	0.25

Case 1. Constant flexural rigidity, line load, no end load

Under these assumptions, the differential equation and boundary conditions become

$$D\frac{d^4w}{dx^4} + (\rho_m - \rho_w)gw(x) = V_o\delta(x) \qquad (2)$$

$$\lim_{|x|\to\infty} w(x) = 0 \qquad \text{and} \qquad \lim_{|x|\to\infty}\frac{dw}{dx} = 0$$

Take the fourier transform of the differential equation where the forward and inverse transforms are defined as

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}dk \qquad (3)$$

where the wavenumber is now $2\pi/\lambda$ instead of the usual $1/\lambda$. I have switched notation because it saves writing 2π many times and also these are old notes. The derivative property is now $\Im[dw/dx] = ik\,\Im[w]$. The fourier transform of the differential equation is

$$Dk^4 W(k) + (\rho_m - \rho_w) g W(k) = V_o \tag{4}$$

and the solution for plate deflection is simply

$$W(k) = \left[k^4 + \frac{4}{\alpha^4}\right]^{-1} \frac{V_o}{D} \tag{5}$$

where the flexural parameter α is (see Turcotte and Schubert, equation (3-127)).

$$\alpha^4 = \frac{4D}{g(\rho_m - \rho_w)} \tag{6}$$

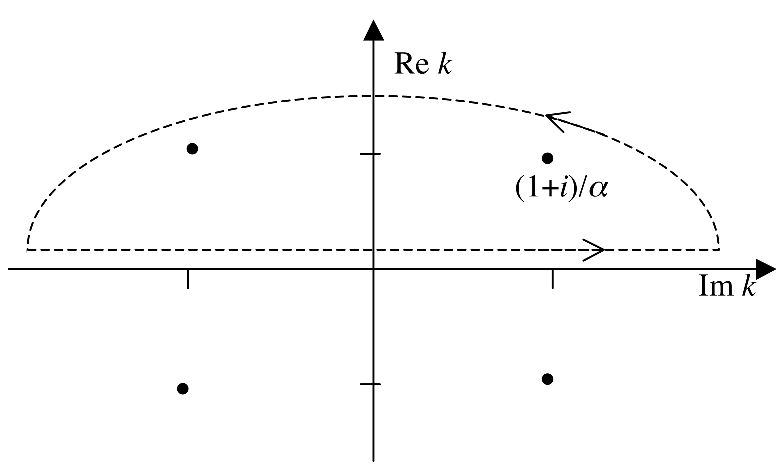
Now take the inverse fourier transform of equation (6).

$$w(x) = \frac{V_o}{2\pi D} \int_{-\infty}^{\infty} \frac{e^{ikx}}{\left(k^4 + \frac{4}{\alpha^4}\right)} dk \tag{7}$$

As in the other solutions, we find the poles in the denominator of (7) and integrate around the poles.

$$\left(k^4 + \frac{4}{\alpha^4}\right) = \left(k^2 + \frac{2i}{\alpha^2}\right)\left(k^2 - \frac{2i}{\alpha^2}\right) \tag{8}$$

$$\left(k^4 + \frac{4}{\alpha^4}\right) = \left(k - \frac{1+i}{\alpha}\right)\left(k - \frac{-1+i}{\alpha}\right)\left(k - \frac{1-i}{\alpha}\right)\left(k - \frac{-1-i}{\alpha}\right)$$



First consider the case for $x > 0$. To match the boundary conditions at infinity we want $\mathrm{Im}(k) > 0$. Thus we close the integration in the upper half of the plane and apply the Cauchy Residue Theorem

$$\oint \frac{f(z)}{z - z_o} dz = i2\pi f(z_o) \tag{9}$$

The relevant poles are

$$k = \frac{1+i}{\alpha} \qquad \text{and} \qquad k = \frac{-1+i}{\alpha} \tag{10}$$

The solution is

$$w(x) = \frac{V_o}{2\pi D} 2\pi i \left[\frac{\alpha^3 e^{i\left(\frac{1+i}{\alpha}\right)x}}{(1+i+1-i)(1+i-1+i)(1+i+1+i)} + \frac{\alpha^3 e^{i\left(\frac{-1+i}{\alpha}\right)x}}{(-1+i-1-i)(-1+i-1+i)(-1+i+1+i)} \right] \tag{11}$$

After some simplification this becomes

$$w(x) = \frac{V_o \alpha^3}{8D} e^{-x/\alpha} \left[\frac{e^{ix/\alpha}}{(1+i)} + \frac{e^{-ix/\alpha}}{(1-i)} \right] \tag{12}$$

This can be re-written in terms of $\cos(x/\alpha)$ and $\sin(x/\alpha)$. Also we know that the solution should be symmetric about $x = 0$. The final result for positive x matches equation (3-130) in Turcotte and Schubert.

$$w(x) = \frac{V_o \alpha^3}{8D} e^{-x/\alpha} \left[\cos(x/\alpha) + \sin(x/\alpha) \right] \tag{13}$$

The important parameters and length scales in this solution are

α	flexural parameter
$2\pi\alpha$	flexural wavelength
$x_o = 3\pi\alpha/4$	distance to the first zero crossing.

The figures on the following page from Turcotte and Schubert display the solution for the continuous plate where the maximum flexure is normalized to 1. In addition the solution for a broken plate is shown. This is also the same form used to model plate bending at a subduction zone. Note for the same downward force, the amplitude of the broken plate is 2 times the amplitude of the continuous plate

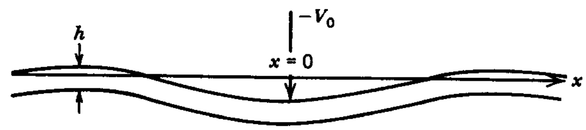


Figure 3-29 Deflection of the elastic lithosphere under a line load.

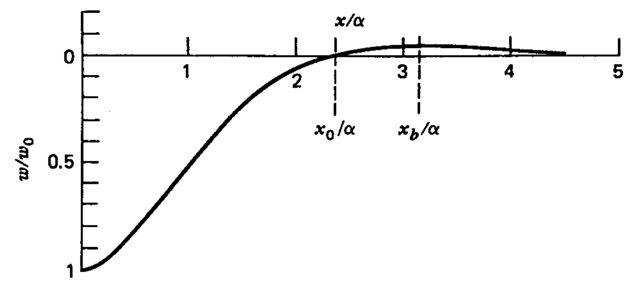


Figure 3-30 Half of the theoretical deflection profile for a floating elastic plate supporting a line load.

$$w = w_0 e^{-x/\alpha} \cos \frac{x}{\alpha} \tag{3-142}$$

This profile is given in Figure 3-32.

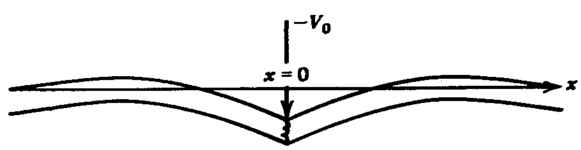


Figure 3-31 Deflection of a broken elastic lithosphere under a line load.

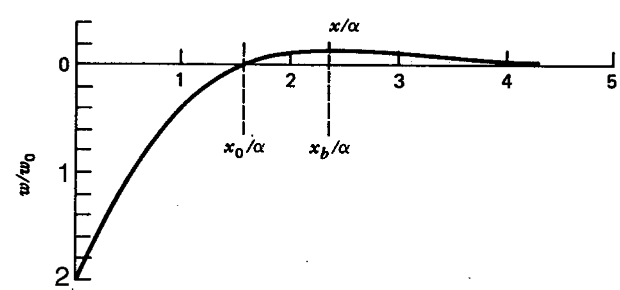


Figure 3-32 The deflection of the elastic lithosphere under an end load.

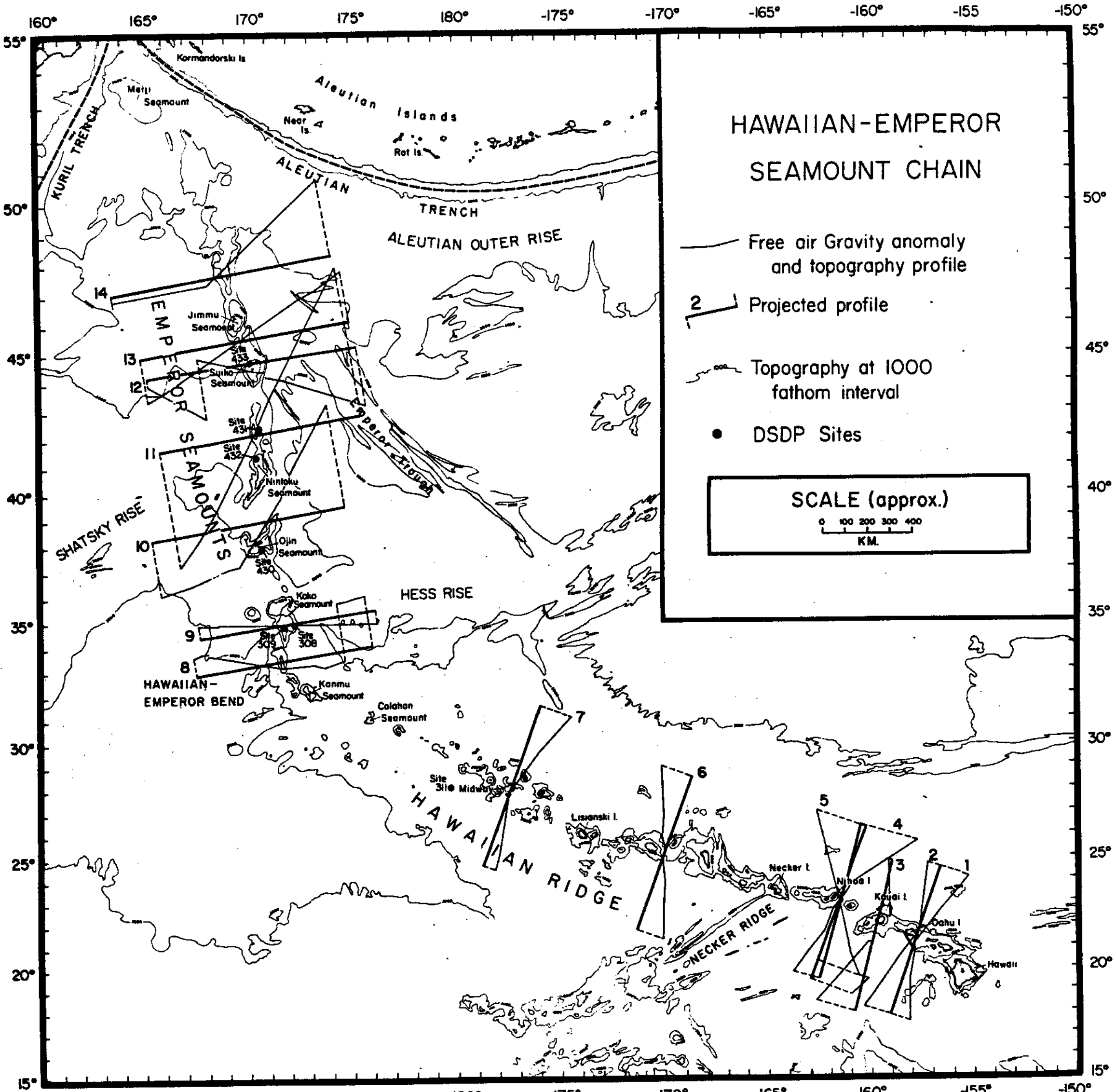
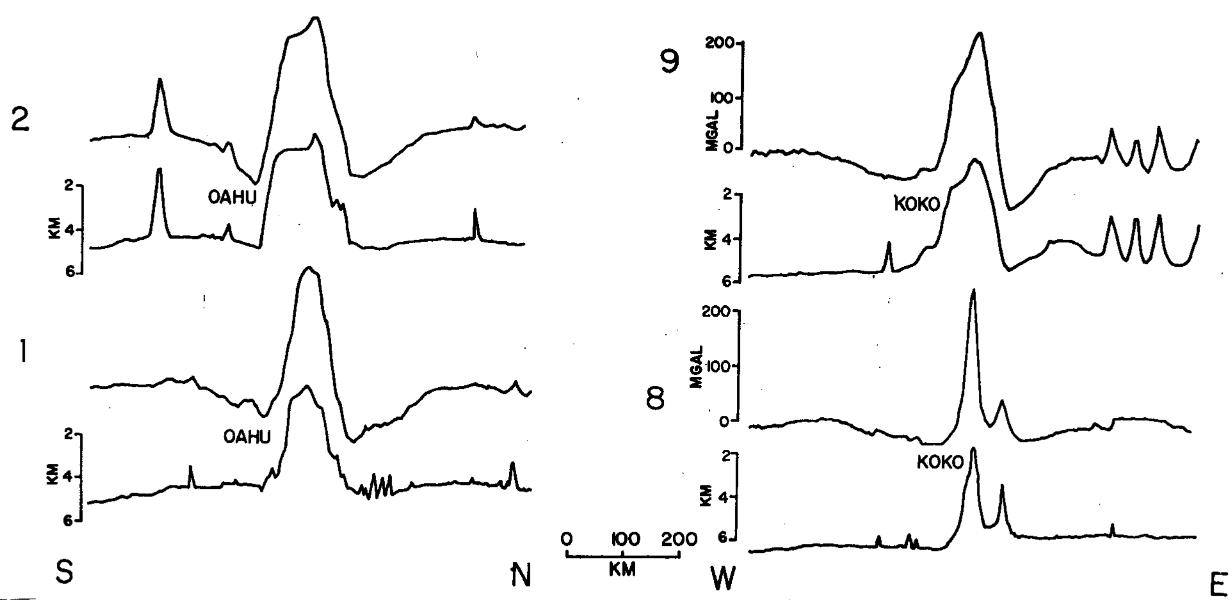


Fig. 1. Location of free air gravity anomaly and bathymetry profiles used in this study. The thin lines indicate the actual ship track, and the heavy lines indicate the profile the data along each track were projected onto. The bathymetry is based on *Chase et al.* [1970]. DSDP sites located on or near the seamount chain are shown as solid circles.



Watts and Cochran, Gravity anomalies and flexure of the lithosphere along the Hawaiian Emperor Chain, Geophys. J. Ror. Astr. Soc., v. 38, p. 119-141, 1979

Case 2. Variable flexural rigidity, arbitrary line load, no end load

For this case we need to solve a linear differential equation but with variabile coefficient. This will involve an iteration scheme in the fourier transform domain where the first iteration is basically equation (5) above. See the original derivation in *Sandwell* [1984] (Thermomechaical Evolution of Oceanic Fracture Zones, *J. Geophys. Res., v. 89,* p. 11401-11413, 1984.) The differential equation and boundary conditions are

$$\frac{d^2}{dx^2}\left(D(x)\frac{d^2 w(x)}{dx^2}\right) + (\rho_m - \rho_w)gw(x) = P(x) \tag{14}$$

$$\lim_{|x|\to\infty} w(x) = 0 \qquad \text{and} \qquad \lim_{|x|\to\infty} \frac{dw}{dx} = 0$$

where *D(x)* is now the spatially variable flexural rigidity, *w(x)* is the deflection of the plate and *P(x)* is the applied load. It is assumed that *D, w,* and *P* are band-limited functions so their fourier transforms exist. The functions *D* and *w* can be written as

$$D(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} D(s)e^{isx}ds \tag{15}$$

$$w(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(r)e^{irx}dr$$

Upon substitution of these expressions for *D* and *w* into the first term of equation (14) and differentiating under the integral, the following is obtained

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(r+s)^2 r^2 D(s)W(r)e^{i(s+r)x}drds + g(\rho_m - \rho_w)w(x) = P(x) \tag{16}$$

The fourier transform of (16) is

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(r+s)^2 r^2 D(s)W(r)\int_{-\infty}^{\infty} e^{i(s+r-k)x}dxdrds + g(\rho_m - \rho_w)W(k) = P(k) \tag{17}$$

By making use of the definition of the delta function,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(s+r-k)x}dx = \delta\left[r-(k-s)\right] \tag{18}$$

and performing the integral with respect to *r*, and using the band-limited property of *D(s)* (i.e., $D(s) = 0\ |s| > \beta$) , equation (17) reduces to a Fredholm integral equation

$$\frac{k^2}{2\pi}\int_{-\beta}^{\beta} D(s)W(k-s)(k-s)^2 ds + g(\rho_m - \rho_w)W(k) = P(k) \tag{19}$$

Notice that when the flexural rigidity is constant $D(x) = D_o$ then $D(s) = 2\pi D_o\delta(s)$. For this case, the solution for the plate deflection for an arbitrary load is

$$W(k) = \left[D_o k^4 + g(\rho_m - \rho_w)\right]^{-1} P(k) \tag{20}$$

Now consider the more general case of spatially variable flexural rigidity

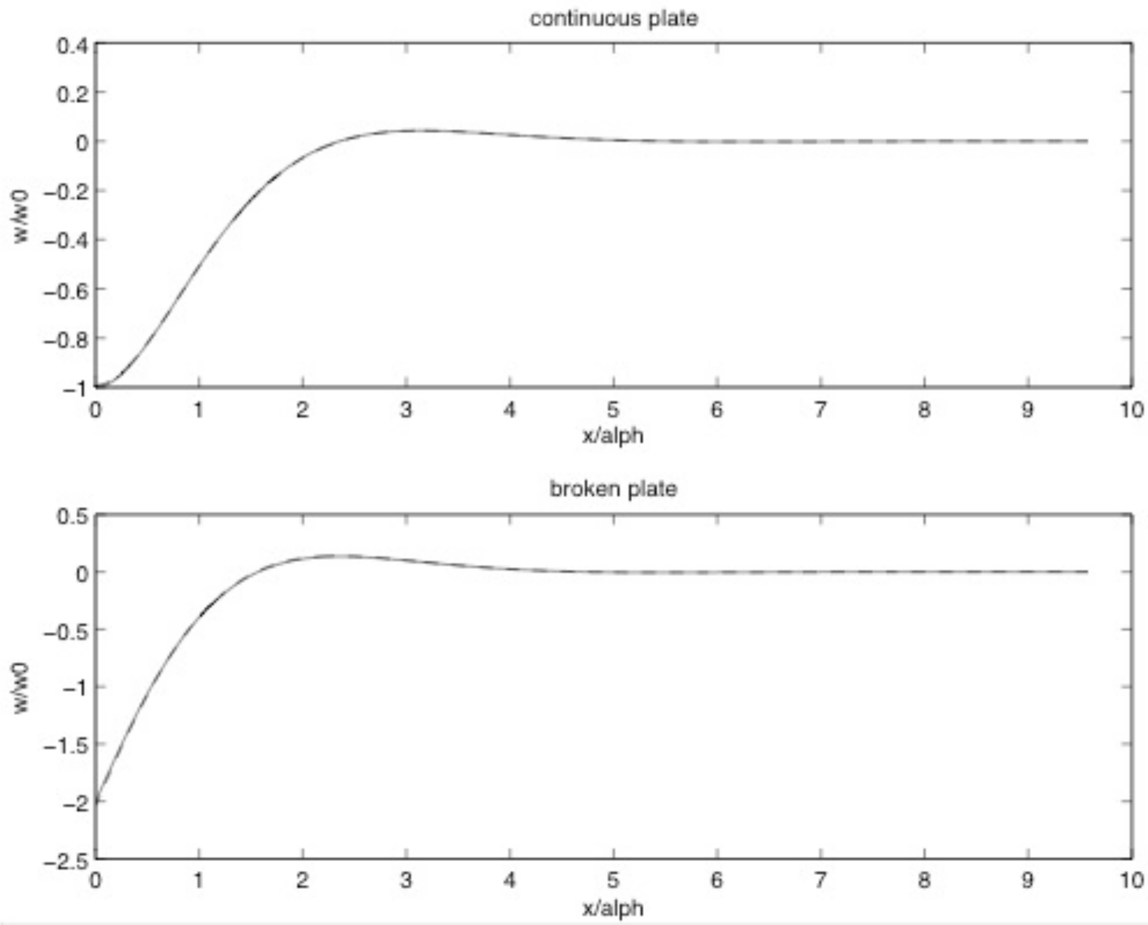
$$D(s) = D'(s) + 2\pi D_o\delta(s) \tag{21}$$

Inserting equation (21) into equation (19) and rearranging terms yields

$$W(k) = \left[D_o k^4 + g(\rho_m - \rho_w)\right]^{-1}\left[P(k) - \frac{k^2}{2\pi}\int_{-\beta}^{\beta} D'(s)W(k-s)(k-s)^2 ds\right] \tag{22}$$

The plate deflection appears on both sides of equation (22) so there is no closed form solution for *W(k)*. However, if the variations in flexural rigidity D' are small compared with the mean value of flexural rigidity D_o, then this equation can be solved by successive approximation. The original derivation in *Sandwell* [1984] provides the necessary requirement for convergence but a numerical illustration is also useful.

The figure below is a numerical example of flexure of a plate with a sharp reduction in plate thickness at the origin. The upper curve compares the flexure of a continuous plate, (continuous curve - analytic solution, equation 13) to the Fourier transform solution to equation 2 (dashed curve). The lower plot is a comparison of the analytic solution to flexure of a broken plate (continuous curve - T&S, equation 3-140) to the numerical iterative solution of equation (22) (dashed curve). For this case the thickness of the plate at the origin was reduced by 95%. This approximates the broken plate solution.


continuous plate
w/w0
x/alph
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0 1 2 3 4 5 6 7 8 9 10
broken plate
w/w0
x/alph
0.5
0
−0.5
−1
−1.5
−2
−2.5
0 1 2 3 4 5 6 7 8 9 10