ELASTIC SOLUTIONS FOR STRIKE-SLIP FAULTING

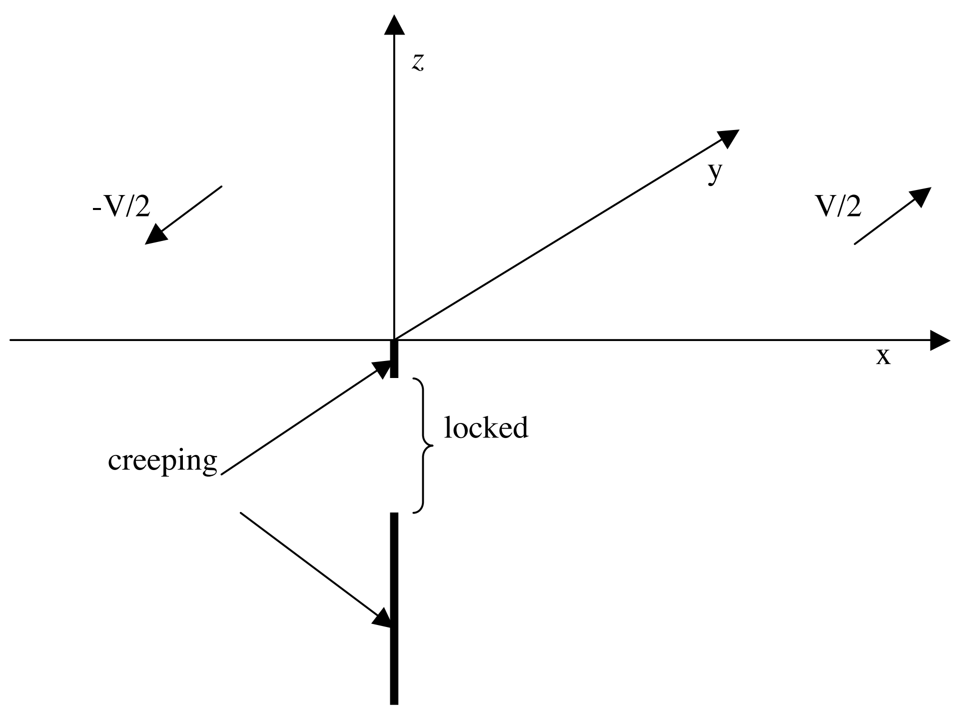
(Reference: Cohen, S. C., Numerical Models of Crustal Deformation in Seismic Zones, *Advances in Geophysics, v. 41*, p. 134-231)

Today's lecture will be the mathematical development of the strain pattern due to strike-slip deformation on a partially locked fault. The notes come from Chapter 8 of Turcotte & Schubert but I'll focus on section 8-6 through 8-9. The section on the North Anatolian Fault is particularly relevant given the recent August and November, 1999 earthquakes which killed more than 15,000 people as many poorly-constructed buildings collapsed.

While I'll follow the overall theme of Chapter 8, I'll deviate in two respects. First I'll use a co-ordinate system with the z-axis pointed upward to be consistent with my previous notes on gravity, magnetics and heat flow. Second I'll develop the solution using fourier transforms to be consistent with my previous notes.

Interseismic Strain Buildup

The first objective is to derive an expression for the surface displacement $v(x)$ and surface strain $\delta v/\delta x$ for the model shown below. A constant velocity V is applied at an elastic half space. There is a fault in the half space with locked and creeping section. Because of free slip on the faults, the strain will be concentrated near the fault.

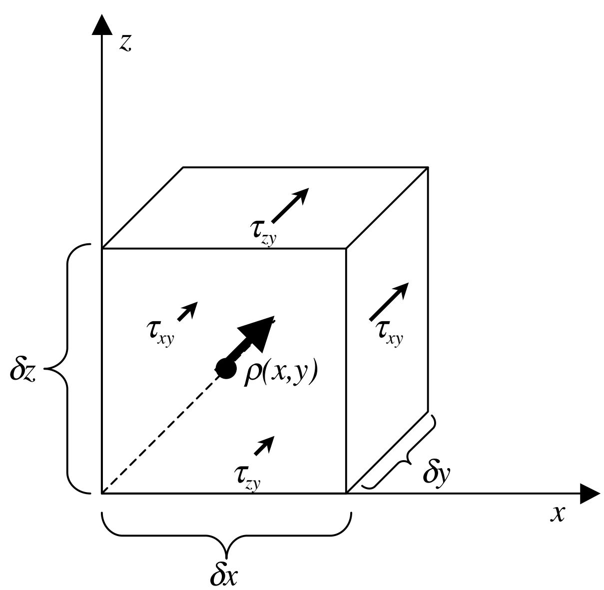


The approach will be as follows:

I. *Develop the force balance from basic principles.*
II. *Establish the line-source Green's function for an elastic full space.*
III. *Establish the screw-dislocation Green's function for an elastic full space.*
IV. *Use the method of images to construct a half-space solution.*
V. *Integrate the line sources to develop the solutions found in the literature.*
VI. *Inclined fault plane*
VII. *Matlab examples*

I. Force Balance

Consider the forces acting on the infinitely-long square rod depicted below. The body force per unit volume of rod must be balanced by tractions on the sides of the rod.



The equation for this force balance is

$$\left[\tau_{xy}(x+\delta x)-\tau_{xy}(x)\right]\delta y\delta z+\left[\tau_{zy}(z+\delta z)-\tau_{zy}(z)\right]\delta x\delta y=\rho(x,z)\delta x\delta y\delta z \qquad (1)$$

where τ_{xy} and τ_{zy} are the shear tractions on the side and top of the box, respectively and $\rho(x,y)$ is the body force which depends only on x and z. Dividing through by δx δy δz and taking the limit as all three go to zero, to arrive at:

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{zy}}{\partial z}=\rho(x,z) \qquad (2)$$

Given the following relationship between stress and displacement, the differential equation reduces to Poisson's equation

$$\tau_{xy} = \mu \frac{\partial v}{\partial x} \tag{3}$$

$$\tau_{zy} = \mu \frac{\partial v}{\partial z}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\mu}\rho(x,z) \tag{4}$$

where μ is the shear modulus and v is the displacement in the y-direction.

II. Line-Source Green's Function

We can generate the solution to an arbitrary source distribution by first developing the line-source Green's function. Consider a line source at a depth of $-a$. The differential equation is:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} = \frac{A}{\mu}\delta(\mathrm{x})\delta(z+a) \tag{5}$$

where A is the source strength having units of force/length. The boundary conditions for this second-order, partial differential are that v must vanish as both $|x|$ and $|z|$ go to infinity. The 2-dimensional forward and inverse fourier transforms are defined as

$$F(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x}) e^{-i2\pi(\mathbf{k}\bullet\mathbf{x})} d^2\mathbf{x} \tag{6}$$

$$f(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mathbf{k}) e^{i2\pi(\mathbf{k}\bullet\mathbf{x})} d^2\mathbf{k}$$

where $\mathbf{k}=(k_x, k_y)$ and $\mathbf{x} = (x, y)$. Take the 2-dimensional fourier transform of the differential equation (5).

$$-(2\pi)^2\left(k_x^2 + k_z^2\right)V(\mathbf{k}) = \frac{A}{\mu}e^{i2\pi k_z a} \tag{7}$$

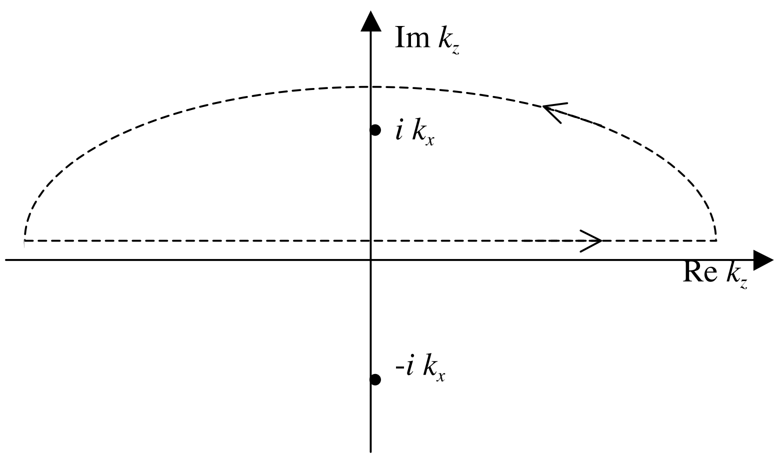
so the solution in the fourier domain is

$$V(\mathbf{k}) = \frac{-Ae^{i2\pi k_z a}}{\mu(2\pi)^2\left(k_x^2 + k_z^2\right)} \tag{8}$$

Now we need to take the inverse fourier transform with respect to k_z and make sure the solution goes to zero as $|z|$ goes to infinity. The integral is

$$V(k_x, z) = \frac{-A}{\mu(2\pi)^2} \int_{-\infty}^{\infty} \frac{e^{i2\pi k_z(z+a)}}{\left(k_x^2 + k_z^2\right)} dk_z \qquad (9)$$

First consider the case $k_x > 0$, z >-a. We can factor the denominator and recognize integrand will vanish for large positive z if we close the contour in the upper hemisphere.



$$V(k_x, z) = \frac{-A}{\mu(2\pi)^2} \oint \frac{e^{i2\pi k_z(z+a)}}{\left(k_z + ik_x\right)\left(k_z - ik_x\right)} dk_z \qquad (10)$$

From the Cauchy integral formula, we know that for any analytic function the following holds for a counterclockwise path surrounding the pole.

$$\oint \frac{f(z)}{z - z_o} dz = i2\pi f(z_o) \qquad (11)$$

In this case with the pole at ik_x, the result is simply

$$V(k_x, z) = \frac{-i2\pi A}{\mu 4\pi^2} \frac{e^{-2\pi k_x(z+a)}}{i2k_x} = \frac{-A}{2\mu} \frac{e^{-2\pi k_x(z+a)}}{2\pi k_x} \qquad (12)$$

Next consider $k_x > 0$, $z < -a$. In this case we must close the integration path in the lower hemisphere to satisfy the boundary conditions; during the integration the only contribution will be from the $-ik_x$ pole. The result will be the same as equation (12). Next consider $k_x < 0$ for $z > -a$ *and* $z < -a$. The z-boundary conditions are met for all of these cases if k_x is replaced by $|k_x|$.

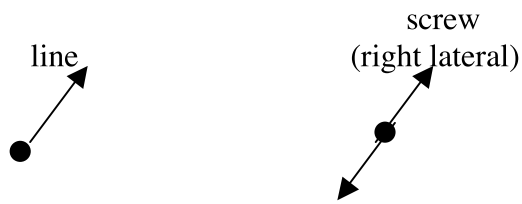
$$V(k_x,z) = \frac{-A}{2\mu}\frac{e^{-2\pi|k_x|(z+a)}}{2\pi|k_x|} \qquad (13)$$

Note this is exactly the same as the gravity solution. The Greens function is the inverse cosine transform of equation (13) or *ln(r)*. The final result is

$$v(x,z) = \frac{-A}{2\pi\mu}\ln\left[x^2 + (z+a)^2\right]^{1/2} \qquad (14)$$

III. Screw Dislocation for Line Source Green's Function

To construct a screw dislocation, one abuts equal but opposite line source dislocations as shown in the diagram below.



A simple way of constructing the screw source is to take the derivative of the line source Green's function in a direction normal to the fault plane. So we need to develop the Green's function for the following differential equation.

$$\nabla^2 v_{screw} = \delta(z+a)\frac{\partial}{\partial x}\delta(x) \qquad (15)$$

To do this we simply take the derivative of the line source Green's function in equation 14.

$$v_{screw}(x,z) = \frac{-A}{2\pi\mu}\frac{\partial}{\partial x}\ln\left\{\left[x^2 + (z+a)^2\right]^{1/2}\right\} = \frac{-Ax}{2\pi\mu}\left[x^2 + (z+a)^2\right]^{-1} \qquad (16)$$

So the Green's function for a line-source screw dislocation at depth is:

$$v_{screw}(x,z) = \frac{-A}{2\pi\mu}\frac{x}{\left[x^2 + (z+a)^2\right]} \qquad (17)$$

IV. Surface boundary condition: Method of images

The surface boundary condition is that the shear stress τ_{zy} must be equal to zero.

$$v(x,z)=\frac{-A}{2\pi\mu}x\left\{\left[x^2+(z+a)^2\right]^{-1}+\left[x^2+(z-a)^2\right]^{-1}\right\}$$

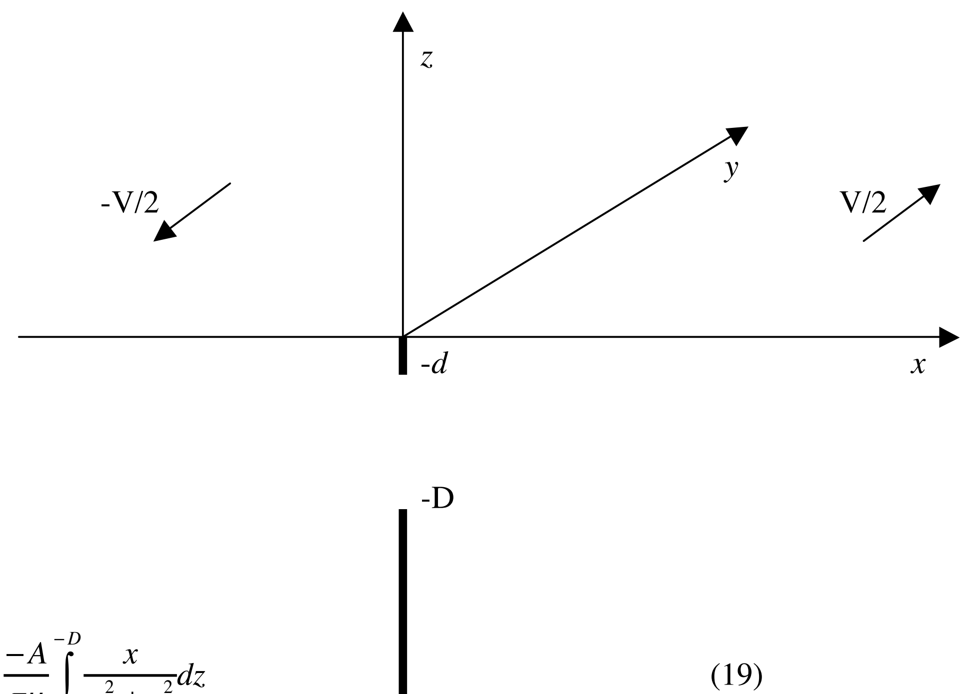
$$v(x,0)=\frac{-A}{\pi\mu}\frac{x}{\left[x^2+a^2\right]} \tag{19}$$

This boundary condition will be satisfied if we place an image source at $z = a$. The effect is simply to double the strength of the Green's function.

V. *Vertical integration of line source to create a fault plane*

The final step in the development is to integrate the line-source screw dislocation over depth.

Case 1. First consider a fault that is free-slip between a depth *-D* and infinity. This is the solution considered by Savage [Savage, J. C., Equivalent strike-slip cycles in half-space and lithosphere-asthenosphere earth models, *J. Geophys. Res., v. 95*, p. 4873-4879, 1990.].



$$v(x)=\frac{-A}{\pi\mu}\int_{-\infty}^{-D}\frac{x}{x^2+z^2}dz \tag{19}$$

To integrate (19) make the following substitution

$$\eta = -xz^{-1} \quad \text{so} \quad d\eta = xz^{-2}dz \tag{20}$$

The integral becomes

$$v(x) = \frac{-A}{\pi\mu}\int_0^{x/D}\frac{1}{1+\eta^2}d\eta \quad = \quad \frac{-A}{\pi\mu}\tan^{-1}\left(\frac{x}{D}\right) \tag{21}$$

We know that $v(\pm\infty) = \pm V/2$ so $A = -V\mu$. Note that A has units of force per unit area times a velocity. This correcponds to a moment rate per area of fault. The familiar results for displacement and shear stress are

$$v(x) = \frac{V}{\pi}\tan^{-1}\frac{x}{D}$$

$$\tau_{xy} = \frac{\mu V}{\pi D}\frac{1}{1+\left(\frac{x}{D}\right)^2} \tag{22}$$

Consider the extreme cases of a completely unlocked fault so D=0. The displacement field will be a step function and the stress will be everywhere zero except at the origin where it will be infinite.

Case 2. Next consider a fault that is free-slip between the surface and a depth $-d$. In this case the integral is

$$v(x) = \frac{V}{\pi}\int_{-x/a}^{0}\frac{1}{1+\eta^2}d\eta \quad = \quad \frac{V}{\pi}\tan^{-1}\eta \Big|_{-x/d}^{x/0} \tag{23}$$

There are two cases depending on whether x is positive or negative.

$$v(x) = \frac{V}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{x}{d}\right) \quad x > 0 \tag{24}$$

$$v(x) = \frac{V}{\pi}\left(\frac{-\pi}{2} - \tan^{-1}\frac{x}{d}\right) \quad x < 0$$

By combining these, the displacement and shear stress are

$$v(x) = \mathrm{V}[H(x) - 1/2] - \frac{\mathrm{V}}{\pi}\tan^{-1}\frac{x}{d}$$

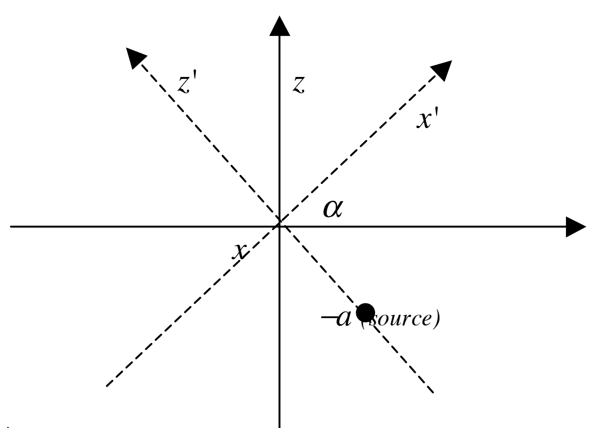
$$\tau_{xy} = \mu \mathrm{V}\left[\delta(x) - \frac{1}{\pi d}\frac{1}{1+\left(\frac{x}{d}\right)^2}\right] \qquad (25)$$

If the fault is completely unlocked so a goes to infinity, the displacement becomes a step and the shear stress is infinite at the origin in agreement with our concepts of a free-slipping fault.

VI. Inclined fault plane

Now consider a model where the fault plane is not perpendicular to the free surface of the earth. The angle α between the vertical and the fault plane will introduce an asymmetry in the model. Later we'll consider the effect on the surface heat flux.

To develop this solution we'll start with the surface displacement due to a screw dislocation. We'll integrate over depth and rotate from the inclined frame into the horizontal frame. Finally we'll introduce the image source to reconcile the free surface boundary condition. From equation 17 we have.



$$v(x', z') = \frac{-A}{2\pi\mu}\frac{x'}{\left[x'^2 + \left(z' + a\right)^2\right]} \qquad (26)$$

The rotation from the x, z frame to the x', z' frame is

$$x' = x\cos\alpha + z\sin\alpha$$
$$z' = -x\sin\alpha + z\cos\alpha \qquad (27)$$

Also note that D=D'cosα..

As before consider free slip between a depth of -D' and minus infinity.

$$v(x^{'},z^{'}) = \frac{\mathrm{V}}{2\pi}\int_{-\infty}^{-D^{'}} \frac{x^{'}}{x^{'2}+\left(z^{'}+a^{'}\right)^{2}}da^{'} \tag{28}$$

let η = z'+a' so dη= da'.

$$v(x^{'},z^{'}) = \frac{\mathrm{V}}{2\pi}\int_{-\infty}^{-z'-D'} \frac{x^{'}}{x^{'2}+\eta^{2}}d\eta \tag{29}$$

We have performed this integration before (equations 19-20) so it is not repeated here. The result is

$$v(x^{'},z^{'}) = \frac{\mathrm{V}}{2\pi}\tan^{-1}\left(\frac{x^{'}}{D^{'}+z^{'}}\right) \tag{30}$$

To match the surface boundary condition, we introduce an image source extending from +D' to infinity but along an image fault inclined at an angle of -α with respect to the vertical. The displacement from the image is

$$v_{image}(x^{''},z^{''}) = \frac{\mathrm{V}}{2\pi}\tan^{-1}\left(\frac{x^{''}}{D^{''}-z^{''}}\right) \tag{31}$$

Finally combining the source and the image and substituting x and z we find

$$v(x,z) = \frac{\mathrm{V}}{2\pi}\left\{\tan^{-1}\left(\frac{x\cos\alpha+z\sin\alpha}{D^{'}-x\sin\alpha+z\cos\alpha}\right)+\tan^{-1}\left(\frac{x\cos\alpha-z\sin\alpha}{D^{'}-x\sin\alpha-z\cos\alpha}\right)\right\} \tag{32}$$

Now calculate the displacement at z = 0 and substitute D' = D/cosα.

$$v(x) = \frac{\mathrm{V}}{\pi}\tan^{-1}\left(\frac{x\cos^{2}\alpha}{D-x\sin\alpha\cos\alpha}\right) \tag{33}$$

If one plots this solution there are two differences from the vertical strike-slip fault case. First, the displacement pattern is shifted along the x-axis by an amount $D\tan\alpha$. Therefore one can identify a dipping fault by recognizing that the position of the fault based on geodetic measurements is shifted from the position of the fault trace based on field geology.

The second difference is that the solution given in equation (33) solution does not match the far-field boundary conditions of +/- V/2. The hanging wall has more displacement than the foot wall. In the extreme case of a near horizontal fault plane, the hanging wall has the full displacement +V while the foot wall has none. This is to be expected because in our model is driven by a force couple. One can "correct" this asymmetry by subtracting a constant α from the arctangent in (33). It is left as an exercise for the reader to show the final solution is

$$v(x) = \frac{\mathrm{V}}{\pi}\left[\tan^{-1}\left(\frac{x\cos^2\alpha}{D - x\sin\alpha\cos\alpha}\right) - \alpha\right] \tag{34}$$

We see that for α=0, this matches the previous solution, equation 22. Also, we can superimpose several of these solutions to simulate any combination of shallow and deep slip. The stress is the shear modulus times the *x*-derivative of the displacement. After a little algebra one finds.

$$\tau_{xy} = \frac{\mu \mathrm{V}}{\pi D_\alpha}\left[1+\left(\frac{x\cos^2\alpha}{D_\alpha}\right)^2\right]^{-1}\left[\cos^2\alpha + \frac{x\cos^3\alpha\sin\alpha}{D_\alpha}\right] \tag{35}$$

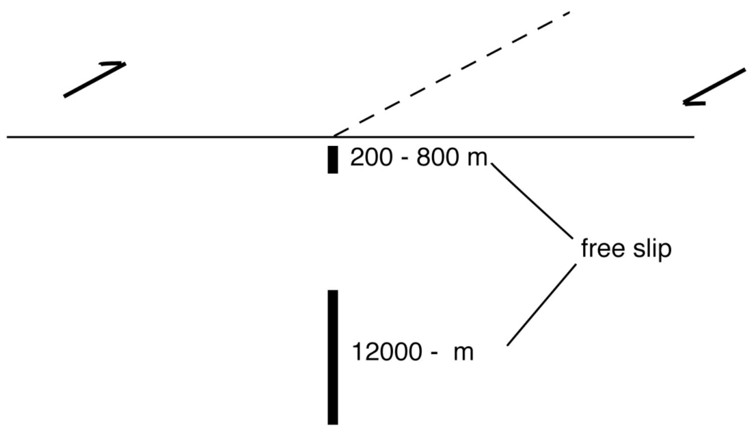
where $D_\alpha = D - x\sin\alpha\cos\alpha$.

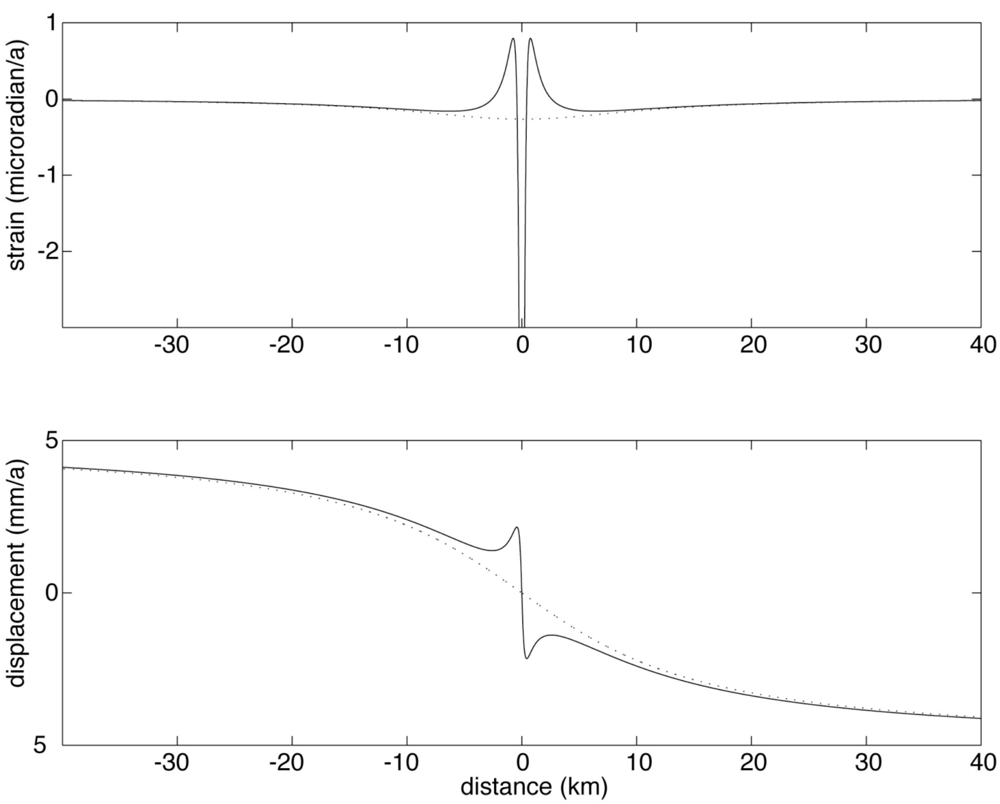
VII. Matlab Examples

The first example is a matlab program to calculate the strain and displacement fields due to a vertical strike-slip fault with free-slip on both shallow and deep fault planes.

```
%
%  program to generate displacement and strain for a screw
%  dislocation. fault slip occurs both shallow and deep.
%
clear
clf
hold off
%%
V=-.01;
D=12000.;
d=800.;
d0=200;
x = -40000:8:40000;
xp = x/1000.;
%
%  this first model has shallow creep between depths of d0 and d
%
v1 = (V/pi)*(atan(x/a0)-atan(x/d));
dv1 = (V/(pi*d0))*1./(1.+(x/d0).^2) - (V/(pi*d))*1./(1.+(x/d).^2);
```

```
%
%  this second model has free-slip for depths greater than D.
%
v2 = (V/pi)*atan(x/D);
dv2 = (V/(pi*D))*1./(1.+(x/D).^2);
%
subplot(2,1,2);plot(xp,(v1+v2)*1000,xp,v2*1000,':');xlabel('distance
(km)');ylabel('displacement (mm/a)')
subplot(2,1,1);plot(xp,1.e6*(dv1+dv2),xp,1.e6*dv2,':');ylabel('strain
(microradian/a)'); axis([-40,40,-3,1])
```





The second example is a matlab program to illustrate the effect of fault dip which simply shifts the arctangent function by an amount $D\tan\alpha$. In this example the shift is 6.9 km.

```
%
%   Compute the displacement due to a dipping fault using equation (34).
%   Note the function atan2() must be used.
%
V=-10;
alph=30*pi/180.;
D=12;
x=-40:40;
%
%
cosa=cos(alph);
sina=sin(alph);
num=x.*cosa*cosa;
dem=D-x.*sina*cosa;
vel0=V*atan2(x,D)/pi;
vel1=V*(atan2(num,dem)/pi-alph/pi);
subplot(2,1,1);plot(x,vel0,x,vel1,'--');
xlabel('distance (km)');ylabel('displacement (mm/a)');
title('dipping 30 degrees in positive x-direction')
grid
%
```