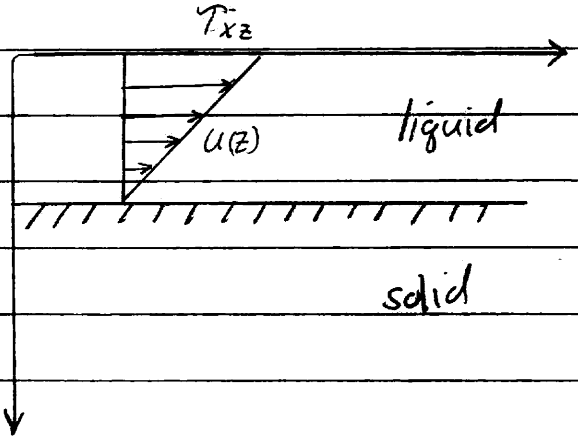
CHAPTER 6 Turcotte & Schubert

Viscosity

$u(z)$ - velocity $m\,s^{-1}$
z - depth m
τ_{xy} - shear stress $N\,m^{-2}$
η - dynamic viscosity $Pa\,S$



strain rate $\dot{\epsilon}$

$$\tau_{xz} = \eta \frac{\partial u}{\partial z}$$ - force balance

$$\nu = \frac{\eta}{\rho}$$ - kinematic viscosity $m^2\,s^{-1}$ (diffusivity)

- Viscosity is a measure of the internal friction opposing deformation.
- Viscosity depends on: temperature, pressure, composition, phase.
- In the above case the stress vs strain relation is linear.
  This is called Newtonian viscosity.

The kinematic viscosity has the same units as the thermal diffusivity so we can form a dimensionless number

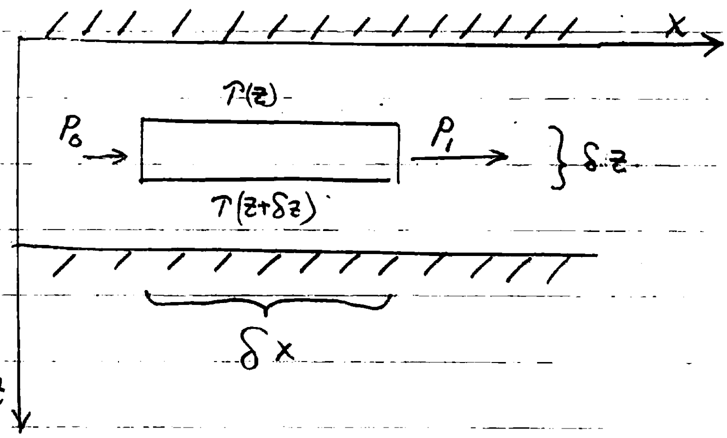
$$P_r = \frac{\nu}{\kappa}$$ Prandtl Number - $\frac{\text{diffusion of momentum}}{\text{diffusion of heat}}$

Transport Properties of common fluids and Earth's mantle
(Table 6-1, T&S)

Table 6-1 Transport Properties of Some Common Fluids at 15°C and Atmospheric Pressure

	Viscosity ~~μ~~ η (Pa s)	Kinematic Viscosity ν ($m^2\ s^{-1}$)	Thermal Diffusivity κ ($m^2\ s^{-1}$)	Prandtl Number Pr
Air	1.78×10^{-5}	1.45×10^{-5}	2.02×10^{-5}	0.72
Water	1.14×10^{-3}	1.14×10^{-6}	1.40×10^{-7}	8.1
Mercury	1.58×10^{-3}	1.16×10^{-7}	4.2×10^{-6}	0.028
Ethyl alcohol	1.34×10^{-3}	1.70×10^{-6}	9.9×10^{-8}	17.2
Carbon tetrachloride	1.04×10^{-3}	6.5×10^{-7}	8.4×10^{-8}	7.7
Olive oil	0.099	1.08×10^{-4}	9.2×10^{-8}	1,170
Glycerine	2.33	1.85×10^{-3}	9.8×10^{-8}	18,880
Mantle	10^{21}	1.8×10^{17}	10^{-6}	1.8×10^{23}

Force Balance in Fluid



$$(P_1 - P_0)\,\delta z = \left[\tau(z+\delta z) - \tau(z)\right]\delta x$$

take limit as $l \to 0$
and $\delta z \to 0$

$$\frac{\partial \tau}{\partial z} = \frac{\partial P}{\partial x} \quad \text{but} \quad \tau = \eta\,\frac{\partial u}{\partial z} \quad \Rightarrow \quad \boxed{\eta\,\frac{\partial^2 u}{\partial z^2} = \frac{\partial P}{\partial x}}$$

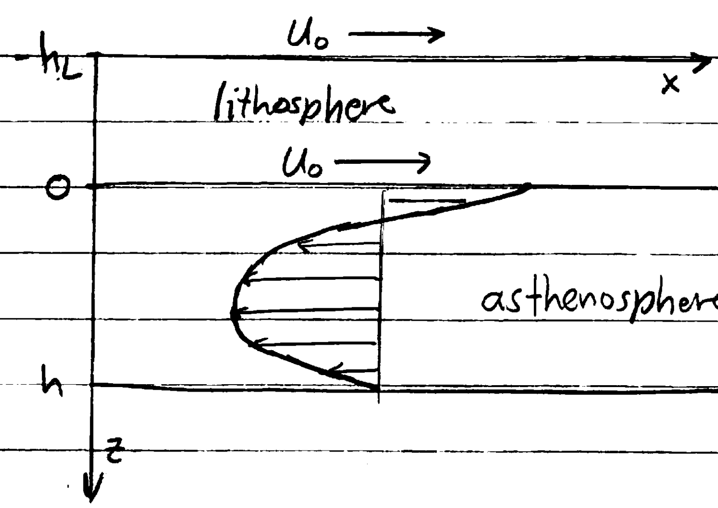
Asthenospheric Counterflow

(Phipps-Morgan, J., W. H.F. Smith, Flattening of the seafloor depth-age curve as a response to asthenospheric flow, Nature, v.359, 524-527, 1992

Suppose we have the simple model illustrated below. A uniform-thickness Pacific plate is sliding across a uniform-thickness asthenospheric channel. Beneath the asthenosphere there is a rigid mantle. Assume the flux of lithosphere is balanced by the flux through the channel.

What is the pressure gradient or drag force on the base of the lithosphere needed to maintain this configuration?

What is the magnitude of the dynamic topography across the basin?



mass conservation

$$u_0 h_L = \int_0^h u(z)\,dz$$

$$u(z) = \frac{1}{2\eta}\frac{\partial P}{\partial x}(z^2 - hz) - u_0\frac{z}{h} + u_0$$

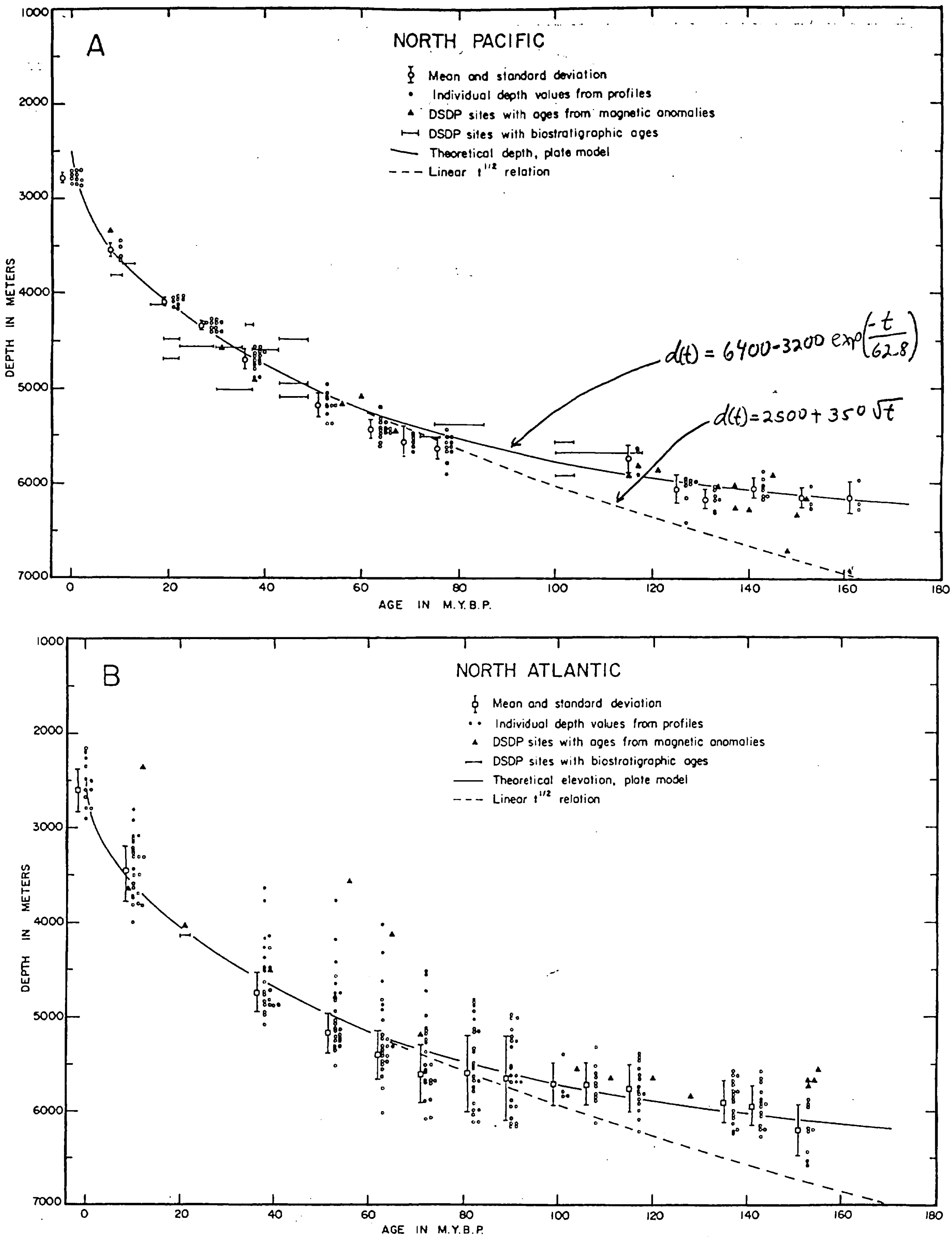
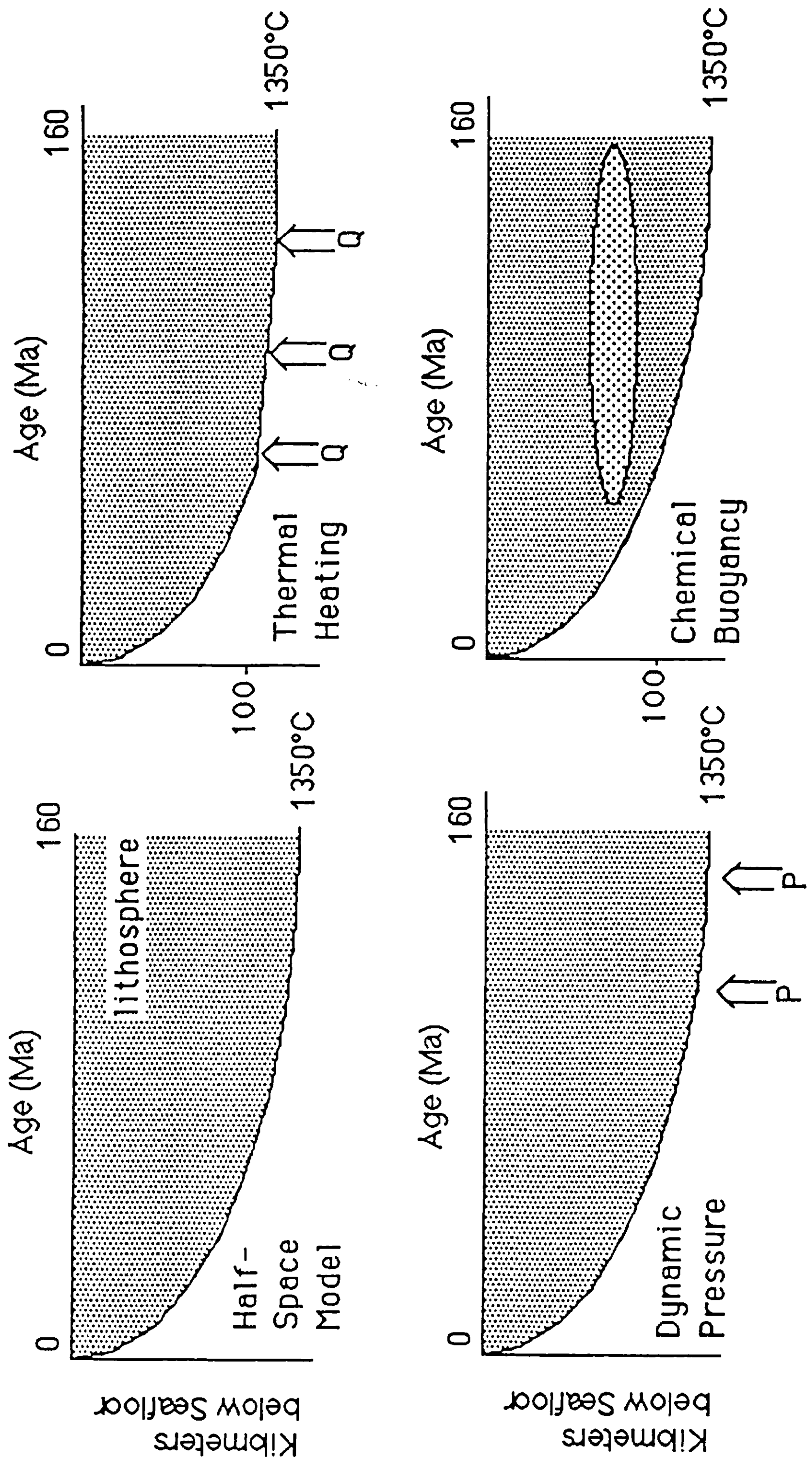


Fig. 13. Plot of individual depth measurements used in calculating mean depths and depths at DSDP sites for (a) the North Pacific and (b) the North Atlantic, illustrating the scatter in depths about the mean value. The means and standard deviations from the individual profile measurements are offset 2 m.y. The correct age is represented by the first column of individual points, some points being slightly offset for clarity. The solid circles in Figure 13b are points from profiles north of 31°N in the data set of Sclater et al. [1975].


Half-Space Model
lithosphere
Thermal Heating
Q
Q
Q
Dynamic Pressure
P
P
Chemical Buoyancy
Age (Ma)
0
160
100
1350°C
Kilometers below Seafloor

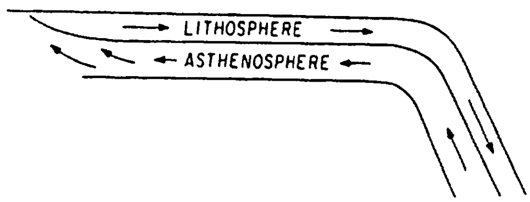


Fig. 1. Model of shallow-mantle convection. Conservation of mass is satisfied by a counterflow in the asthenosphere at depths between 100 and 300 km. The return flow is driven by a hydrostatic pressure that increases with distance from the ridge. This pressure gradient is reflected as an increase in ocean-floor elevation away from the ridge.

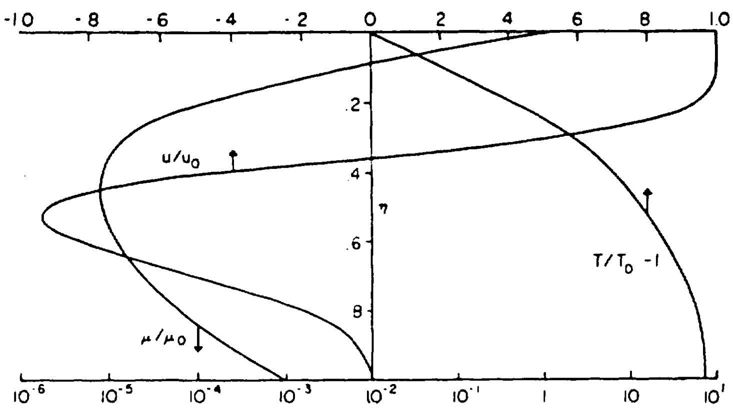


Fig. 2. The dimensionless velocity u/u_0, the dimensionless temperature $(T/T_0) - 1$, and the dimensionless viscosity μ/μ_0 as a function of depth. The parameter values are $L = 600$ km, $u_0 = 1$ cm/year, $\rho = 3.3$ g/cm^3, $k = 7 \times 10^{-3}$ cal/cm sec °K, $E^* = 4.5$ ev, $V_a = 10$ A^3, $V^* = 25$ A^3, $D_0 = 20$ cm^2/sec, and $R_c = 0.2$ cm. The counterflow occurs predominantly in the depth range where the viscosity is minimum.

Conservation Equations for Mantle Convection

1. (Turcotte and Oxburgh, Finite amplitude convection cells, J. Fluid Mech., v.28, p. 29-42, 1967)

2. (Davies, Dynamic Earth, Cambridge University Press, p. 214-237, 1999.)

3. (Schubert, Turcotte and Olson, Mantle Convection in the Earth and Planets, Cambridge Univ. Press, 940 pp, 2001)

- Solution for steady, cellular convection when the Rayleigh number and Prandtl number are large.
- $Ra = 10^6 - 10^8$ $\quad Pr = 10^{24} - 10^{23}$
- Two-dimensional problem
- Boussinesq approximation is valid if temperature is difference between actual and the adiabatic gradient.

Reference State $\quad \rho_0, T_0, P_0 \quad$ adiabatic temperature gradient

Boussinesq Approximation $\quad \rho - \rho_0 = -\rho_0 \alpha (T - T_0)$

let $\Theta = T - T_0 \qquad P = P_0 - \rho_0 g z$

Set of equations that need to be solved for finite amplitude convection. They are non-linear and 3-D

conservation of mass (incompressible)

$$\nabla \cdot \vec{u} = 0$$

conservation of momentum - (Navier - Stokes equation)

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho_0} \nabla p + \nu \nabla^2 \vec{u} + \alpha g \theta \hat{k}$$

acceleration | pressure gradient | viscous resistance | buoyancy z-direction only

conservation of energy

$$\frac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla \theta = \kappa \nabla^2 \theta + \frac{Q}{\rho c_p}$$

time variation in temperature | advection of heat | thermal diffusion | heat generation

Conservation of mass and momentum 2-D

In 2-D when there are no body forces, one can introduce the stream function ψ

$$u = -\frac{\partial \psi}{\partial z} \qquad w = \frac{\partial \psi}{\partial x} \qquad \text{note; } \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = -\frac{\partial^2 \psi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x \partial z} = 0$$

plug ψ into force balance $\boxed{\eta \nabla^2 \vec{u} - \nabla P = 0}$

$$\eta \overbrace{\left(\frac{\partial^3 \psi}{\partial x^2 \partial z} + \frac{\partial^3 \psi}{\partial z^3} \right)}^{\nabla^2 u} + \frac{\partial P}{\partial x} = 0$$

$$\eta \overbrace{\left(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial z^2 \partial x} \right)}^{\nabla^2 w} - \frac{\partial P}{\partial z} = 0$$

take derivative of first w.r.t. z and the second w.r.t. x and add to eliminate pressure.

$$\frac{\partial^4 \psi}{\partial x^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial z^2} + \frac{\partial^4 \psi}{\partial z^4} = 0$$

$\nabla^4 \psi = 0$ biharmonic equation