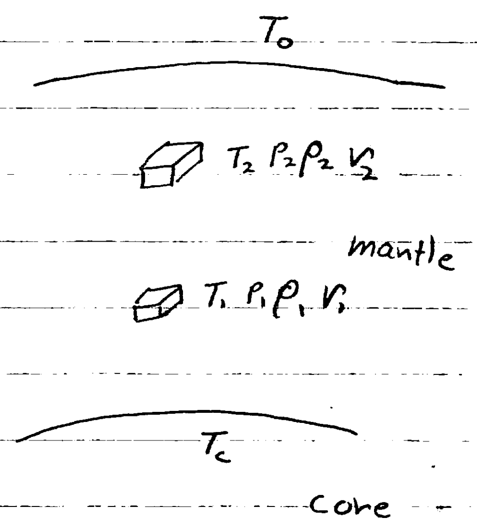
Adiabatic Temperature Gradient

(Stacey, Physics of the Earth, John Wiley & Sons, p. 192-195, 1977)
(Reif, Statistical and Thermal Physics, McGraw-Hill, p. 161-164, 1965)

We would like to know the adiabatic geotherm in the mantle for a starting point on our convection models.



Suppose we have a cube of mantle material and we move this cube from r_1 to r_2 without allowing diffusion of heat across the walls of the cube. The cube will move to an area of lower pressure and thus will affect its density and temperature. This temperature profile is called an adiabat.

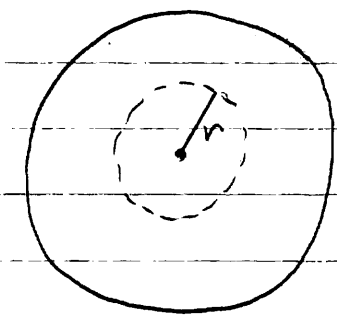
If the temperature gradient exceeds the adiabatic temperature then the layer may begin to convect.
If the temperature gradient is less than the adiabatic gradient then the layer has stable stratification.
The ocean has stable stratification → internal waves
The stratosphere has stable stratification → lee waves

How do we construct the adiabatic geotherm in the mantle?

$$\left(\frac{\partial T}{\partial r}\right)_{adiabatic} = \left(\frac{\partial T}{\partial P}\right)_S \left(\frac{\partial P}{\partial r}\right)$$

T - temperature
P - pressure
S - entropy
V - volume

$$\left(\frac{\partial P}{\partial r}\right) = -g(r)\,\rho(r)$$



$$g(r) = \frac{G\,m(r)}{r^2}$$

$$m(r) = \int_0^r 4\pi r^2 \rho(r)\,dr$$

If we can construct $\rho(r)$ then we can determine $\frac{\partial P}{\partial r}(r)$.

Constraints: a) total mass of earth known from satellite orbit period GM_e

b) moment of inertia known from precession of orbit plane $J_2 = \frac{C-A}{M_e a^2}$

c) normal modes are used to construct the details in the radial density.

Before normal mode methods are used the Adams-Williamson equation to determine $\frac{\partial \rho}{\partial r}$

$$\frac{\partial \rho}{\partial r} = \frac{\partial \rho}{\partial P}\frac{\partial P}{\partial r} = \frac{\rho}{K}\frac{\partial P}{\partial r}$$

K - bulk modulus

$$K = \rho\frac{\partial P}{\partial \rho} \qquad K = \frac{3\lambda + 2\mu}{3}$$

$$V_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$V_p = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$$

V_p - compressional velocity
V_s - shear velocity

$$V_s = \sqrt{\frac{\mu}{\rho}}$$

seismic parameter $\phi = \frac{K}{\rho} = V_p^2 - \frac{4}{3} V_s^2$

$$\boxed{\frac{\partial \rho}{\partial r} = \frac{1}{\phi} \frac{\partial P}{\partial r} = \frac{-g\rho}{\phi}}$$

Adams - Williamson equation

Within the lower mantle and outer core, the radial density structure obeys this equation.

The result is that given the constraints and seismic data one can construct $\rho(r)$

$$\frac{\partial P}{\partial r} = \rho(r) \frac{G}{r^2} \int_0^r 4\pi r^2 \rho(r)\, dr$$

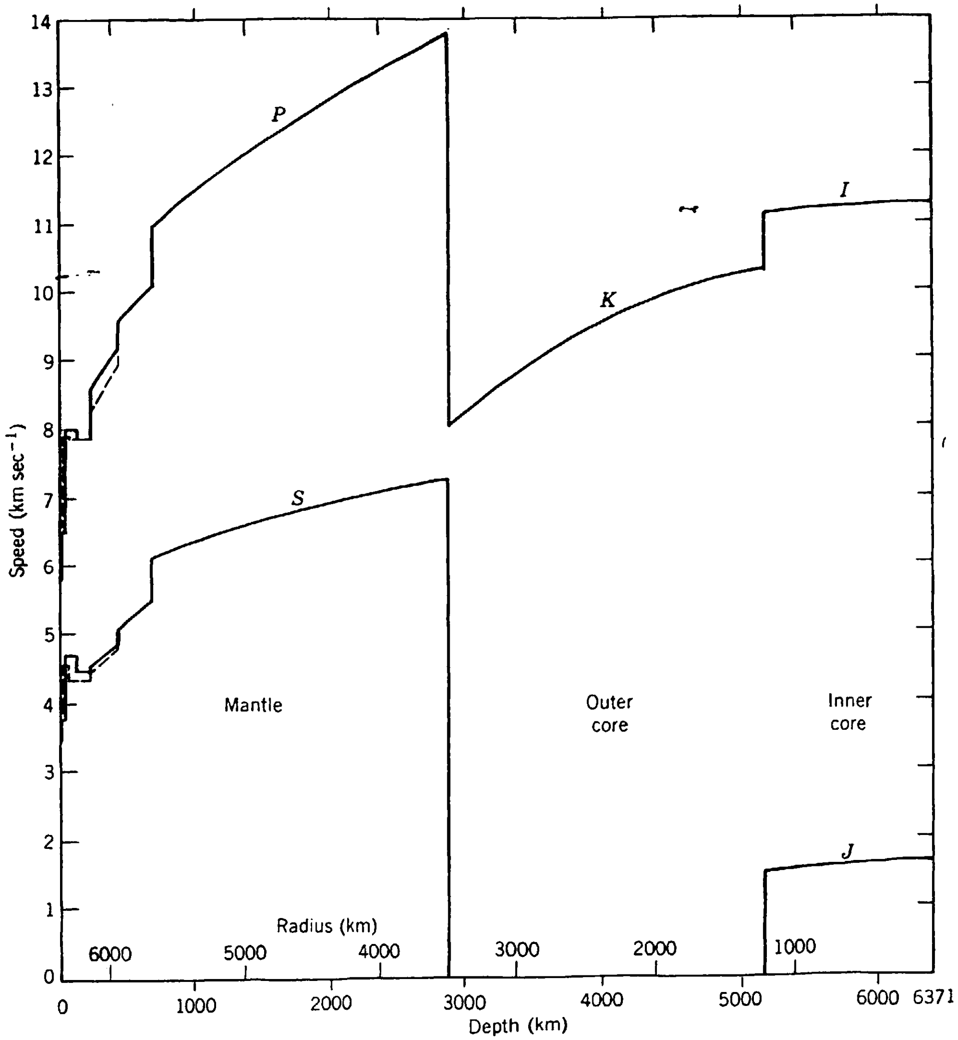


Figure 6.14. Velocities of body waves (P and S) within the Earth. Data for the earth model by Dziewonski et al. (1975)—see Appendix G.

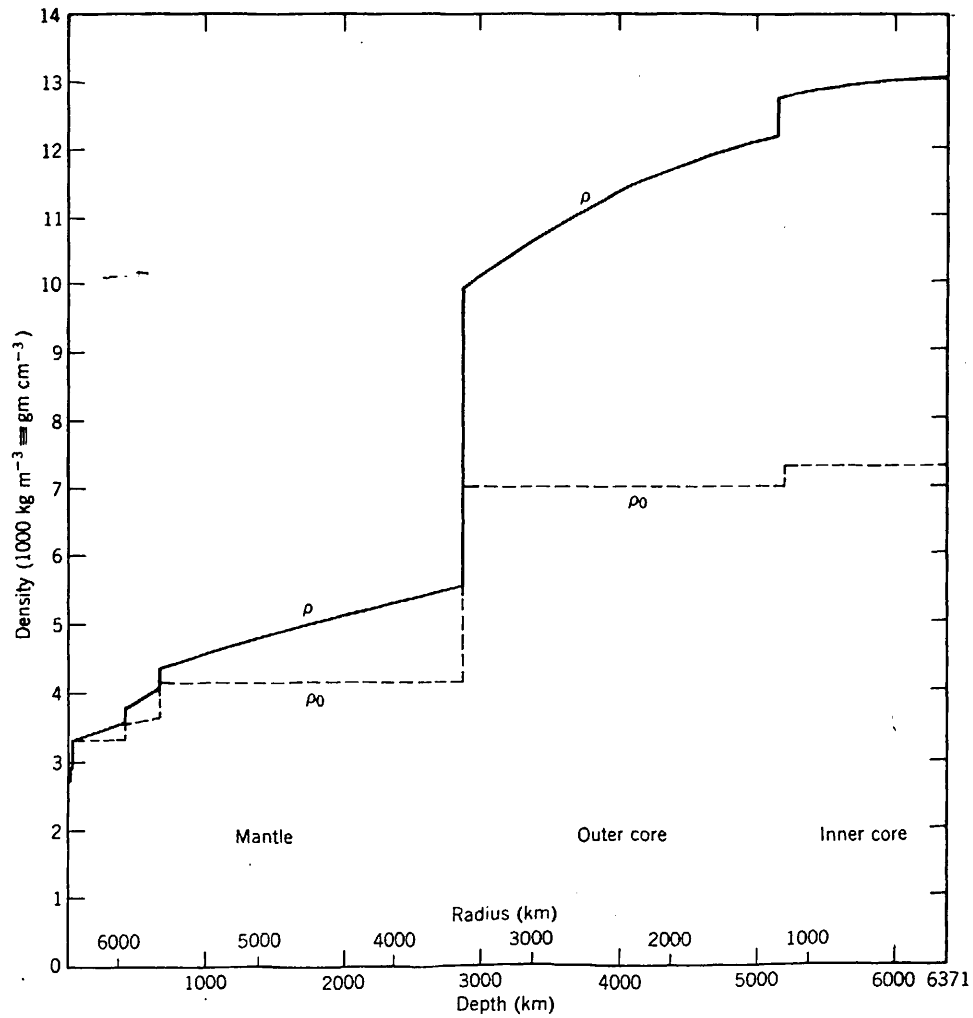


Figure 6.17. Density profile of Earth model by Dziewonski et al. (1975) (solid line) with corresponding extrapolated zero pressure (and room temperature) density (broken line).

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Remember $\left(\frac{\partial T}{\partial r}\right)_{adiabatic} = \left(\frac{\partial T}{\partial P}\right)_S \left(\frac{\partial P}{\partial r}\right)$

just need $\left(\frac{\partial T}{\partial p}\right)_S$

Review of Maxwell relations (Reif, p. 161-164)

First law of thermodynamics: An equilibrium macrostate of a system can be characterized by an internal energy $\bar{E}$. If the system is isolated $\bar{E} = \text{const.}$

$\Delta \bar{E} = -W + Q$

W - work done on cube

Q - heat absorbed by cube

Independent variables S and P.

$P dV = d(PV) - V dP$

$\vdots$

$\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P$

$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial T}{\partial S}\right)_p$$

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

also FYI

$$\alpha = \frac{-1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

$$dS = \frac{dQ}{T} = \frac{mC_p dT}{T} \qquad C_p \equiv \frac{1}{m}\left(\frac{dQ}{dT}\right)_p$$

$$\left(\frac{dT}{dS}\right)_p = \frac{T}{mC_p}$$

$$\left(\frac{\partial T}{\partial P}\right)_S = \frac{V\alpha T}{m\, C_p} = \frac{\alpha T}{\rho C_p}$$

$F = ma$

Now we are done! $\quad J = kg\ m^2 s^{-2}$

$$\left(\frac{\partial T}{\partial r}\right)_{adiabatic} = \frac{\alpha T}{\rho C_p}(-\rho g) = \frac{-\alpha T g}{C_p} \quad \frac{m\, \cancel{s^{-2}}}{\cancel{kg}\, m^2 \cancel{s^{-2}}\, \cancel{kg^{-1}}\, {}^\circ K^{-1}} \quad \frac{{}^\circ K}{m}$$

Example: $g = 9.8\ m s^{-2}$ $\quad T = 1600^\circ K$ $\quad \alpha = 2\times10^{-5}\ {}^\circ K^{-1}$

$C_p = 1200\ J\ kg^{-1}\ {}^\circ K^{-1}$

$$\frac{\partial T}{\partial r} = -.261\ {}^\circ K/km$$

The problem with this expression is that we don't know the radial variation of α and C_p
(Guy says C_p is constant in the mantle ~ 1200)

Need: Grüneisen Parameter γ – This parameter can be measured in the lab and also inferred from fundamental physics $\quad \gamma \sim 1.5$ in the lower mantle

$$\gamma = \frac{\alpha K}{\rho C_p}$$

$$\Phi = \frac{K}{\rho}$$

$$\left(\frac{\partial T}{\partial r}\right) = \frac{-\gamma \rho T g}{K} = \frac{-\gamma T g}{\phi}$$

$$\frac{dT}{T} = -\frac{\gamma g}{\phi}\, dr$$

$$\ln\left(\frac{T_2}{T_1}\right) = \int_{r_1}^{r_2} \frac{\gamma g(r)}{\phi(r)}\, dr \quad \text{also} \quad \frac{T_1}{T_2} \simeq \left(\frac{\rho_1}{\rho_2}\right)^{\gamma}$$

Example: Given temperature of 2000°K at the 660 km seismic discontinuity, what is the temperature at the CMB

$$T_{CMB} = T_{660}\left(\frac{\rho_{CMB}}{\rho_{660}}\right)^{1.5} = 2700°K$$

Guy also says new constraints on the temperature at the top of the outer core are 4000°K. Thus there is a temperature drop of 1300°K at the CMB. This is similar to the 1300°K temperature drop across the lithosphere

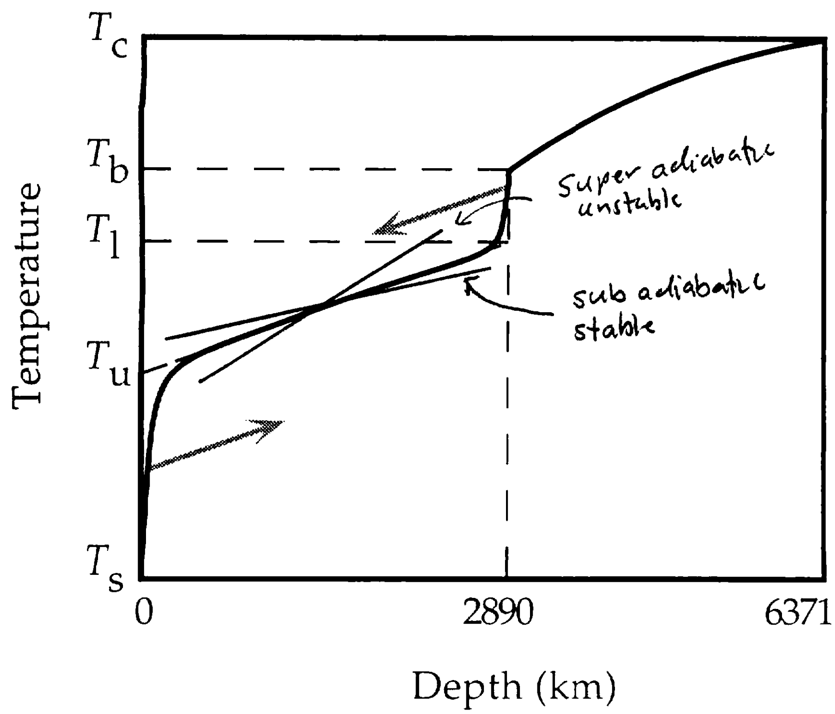

Figure 7.9. Schematic temperature profile through the earth. Thermal boundary layers are assumed at the top of the mantle (the lithosphere) and the bottom of the mantle. Numerical values of the temperatures are quite uncertain (see text). The grey arrows show adiabatic compression and decompression paths of material from the thermal boundary layers.