Conservation Equations for mantle Convection

Schubert, Turcotte and Olson, Mantle Convection, Ch.6.

These are the basic equations relevant to mantle convection. Rocks in the mantle behave as an elastic solid on seismic timescales but as a fluid on geologic timescales.

reference state - Use an equation of state to define an adiabatic temperature gradient.

$\rho = \rho(T, P)$ ρ - density

T - temperature

$\rho_o(z)$, $T_o(z)$, $P_o(z)$ P_o - pressure $\sim \rho g z$

$g(z)$ - acceleration of gravity

The adiabatic temperature gradient in the mantle is the rate of increase in temperature with depth as a result of compression of the rock by the weight of the overlying material. If the element of material is compressed and reduced in volume by increasing pressure, it will also be heated as a result of work done by the pressure force during the compression. If there is no transfer of heat into or out of the element during this process, the compression is said to be adiabatic, and the associated temperature rise is the adiabatic temperature gradient.

over the whole surface, it should be zero if the material is incompressible.

Thus $\nabla \cdot \vec{u} = 0$ by divergence theorem.

perturbations to reference state

$$\rho - \rho_o = -\rho_o \alpha (T - T_o) \qquad \alpha \text{ - thermal expansion coef.}$$

$$\theta = (T - T_o)$$

$$P = P_T - P_o = P_T - \rho g z$$

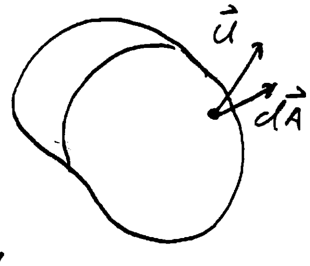
The set of equations that are needed to describe mantle convection are 3-D, time dependent and non-linear

conservation of mass (incompressible)

$$\nabla \cdot \vec{u} = 0 \qquad \vec{u} = (u, v, w) \text{ - velocity vector}$$

divergence theorem $\displaystyle\int_V \nabla \cdot \vec{u}\, dV = \int_A \vec{u} \cdot d\vec{A} = 0$

Consider a volume element V with bounding surface A.
If we integrate the velocity of the fluid normal to the boundary over the whole surface, it should be zero if the material is incompressible.
Thus $\nabla \cdot \vec{u} = 0$ by divergence theorem.



special case of 2-D flow

$\psi(x, y)$ - stream function

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

conservation of momentum (Navier - Stokes equation)

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{\text{acceleration}} + \vec{u} \cdot \nabla \vec{u} = \overbrace{\underbrace{\underbrace{-\frac{1}{\rho} \nabla P}_{\text{pressure gradient}} + \underbrace{\nu \nabla^2 \vec{u}}_{\text{viscous resistance}}}_{\text{post glacial rebound}} + \underbrace{\alpha g \Theta \hat{k}}_{\text{buoyancy}}}^{\text{mantle convection}}$$

accelerations are small in the mantle so LHS $= 0$
coriolis force is small because $|\vec{u}|$ is small

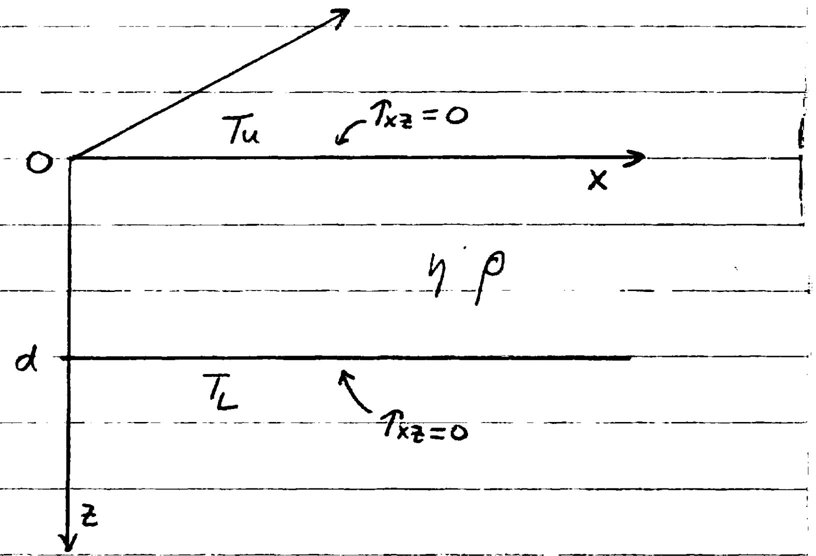
conservation of energy

$$\overbrace{\underbrace{\underbrace{\frac{\partial \Theta}{\partial t}}_{\text{time variation in temp.}} + \underbrace{\vec{u} \cdot \nabla \Theta}_{\text{advection of heat}}^{\text{(non-linear)}} = \underbrace{\kappa \nabla^2 \Theta}_{\text{diffusion of heat}}}_{\text{cooling half space}} + \underbrace{\frac{Q}{\rho C_p}}_{\text{generation of heat}}}^{\text{mantle convection}}$$

Onset of convection and the Rayleigh Number

(T&S, p 267-272)

Start with a fluid layer of viscosity η and density ρ.
Hold the temperature at the top fixed T_u and increase T_L until convection first develops.



What temperature difference is needed?
What is the wavelength of the convection?

Assume free-slip at top and bottom.
Assume 2-D flow.
Incompressible fluid.

Basic State $\quad u_0 = w_0 = 0 \qquad T_0 = T_u + \beta z \qquad \beta = \dfrac{T_L - T_u}{d}$

$$\frac{\partial P_0}{\partial x} = 0 \qquad \frac{\partial P_0}{\partial z} = g\rho$$

Perturbation $\quad T = T_0 + \theta$

$\rho = \rho_0(1 - \alpha\theta)$ Boussinesq approximation

$P = P_0 + \delta P$

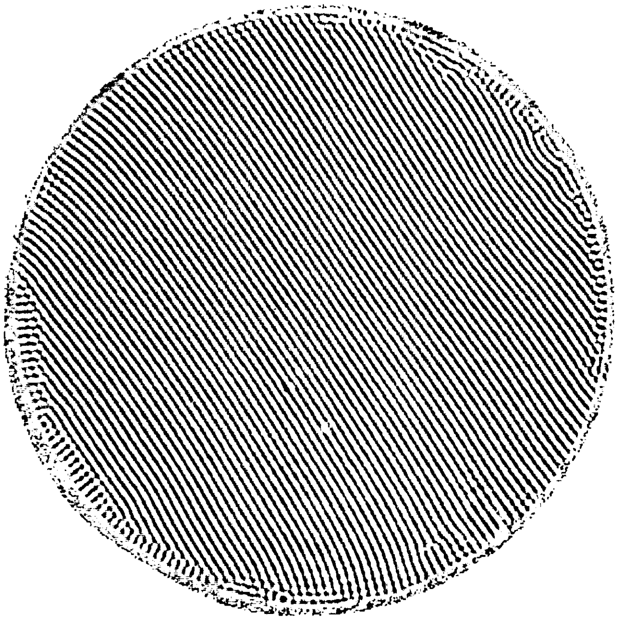
u, w these are small

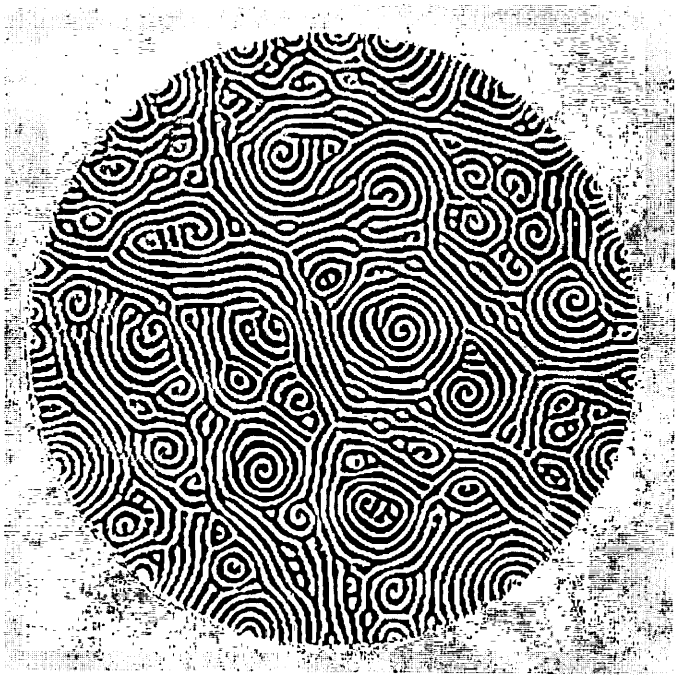
Bénard-Marangoni Convection in Two Layered Liquids

Physical Review Letters, **84**, 3590 (2000).

Wayne A. Tokaruk, T. C. A. Molteno and Stephen W. Morris
Department of Physics, University of Toronto, 60 St. George St., Toronto, Ontario, Canada M5S 1A7.

We describe experiments on Bénard-Marangoni convection in horizontal layers of two immiscible liquids. Unlike previous experiments, which used gases as the upper fluid, we find a square planform close to onset which undergoes a secondary bifurcation to rolls at higher temperature differences. The scale of the convection pattern is that of the thinner lower fluid layer for which buoyancy and surface tension forces are comparable. The wavenumber of the pattern near onset agrees with the prediction of the linear stability analysis for the full two-layer problem. The square planform is in qualitative agreement with recent one- and two-layer nonlinear theories, which fail however to predict the transition to rolls.





mass $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$

momentum

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho_0} \nabla P + \frac{\eta}{\rho_0} \nabla^2 \vec{u} + \alpha g \Theta \hat{k}$$

energy $\frac{\partial \Theta}{\partial t} + \vec{u} \cdot \nabla \Theta = \kappa \nabla^2 \Theta$

Terms with products of small quantities can be neglected. Also assume marginal stability so the acceleration terms are small (infinite Pr) but the perturbation will grow or die. Convective rolls develop at marginal instability so use 2-D.

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\eta \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\partial p}{\partial x} = 0$$

$$\eta \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{\partial p}{\partial z} - \alpha g \rho_0 \Theta = 0$$

$$\kappa \left(\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial z^2} \right) - w\beta = \frac{\partial \Theta}{\partial t}$$

introduce stream function

$$u = -\frac{\partial \psi}{\partial z} \qquad w = \frac{\partial \psi}{\partial x}$$

take x-derivative of z equation and z-derivative of x-equation and add

$$\eta\left(\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial z^2} + \frac{\partial^4 \psi}{\partial z^4}\right) - \rho_0 \alpha g \frac{\partial \theta}{\partial x} = 0$$

$$k\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2}\right) = \beta \frac{\partial \psi}{\partial x} + \frac{\partial \theta}{\partial t}$$

B.C. $\theta(0) = \theta(d) = 0$ $\qquad \frac{\partial \psi}{\partial x} = 0 \quad z = 0, d$ (zero vertical velocity)

$\frac{\partial^2 \psi}{\partial z^2} = 0 \quad z = 0, d$ (zero shear stress)

Assume solutions of the form.

$$\psi = \psi_0 \sin\left(\frac{\pi z}{d}\right) \sin(2\pi k x)\, e^{\alpha' t} \qquad \frac{\partial^2 \psi}{\partial z^2} = -\left(\frac{\pi}{d}\right)^2 \psi \qquad \frac{\partial^2 \psi}{\partial x^2} = -(2\pi k)^2 \psi$$

$$\theta = \theta_0 \sin\left(\frac{\pi z}{d}\right) \cos(2\pi k x)\, e^{\alpha' t} \qquad \frac{\partial^2 \theta}{\partial z^2} = -\left(\frac{\pi}{d}\right)^2 \theta \qquad \frac{\partial^2 \theta}{\partial z^2} = -(2\pi k)^2 \theta$$

$\alpha' > 0$ unstable $\qquad \alpha' < 0$ stable

$$\eta\left[(2\pi k)^2+\left(\frac{\pi}{d}\right)^2\right]^2 \Psi_0 = -\rho_0 \alpha g (2\pi k)\,\Theta_0 \qquad \text{(momentum)}$$

$$\left[\alpha' + \kappa\left(\frac{\pi}{d}\right)^2 + \kappa(2\pi k)^2\right]\Theta_0 = -2\pi k\,\beta\,\Psi_0 \qquad \text{(energy)}$$

from the first equation

$$\Psi_0 = \frac{-\rho_0 \alpha g (2\pi k)}{\eta}\,\Theta_0\left[(2\pi k)^2+\left(\frac{\pi}{d}\right)^2\right]^{-2}$$

plus into second equation

$$\left[\alpha' + \kappa\left(\frac{\pi}{d}\right)^2 + \kappa(2\pi k)^2\right]\cancel{\phi_0} = \frac{(2\pi k)^2 \rho_0 \alpha g \beta}{\eta}\,\cancel{\phi_0}\left[(2\pi k)^2+\left(\frac{\pi}{d}\right)^2\right]^{-2}$$

$$\alpha' = \frac{\rho_0 \alpha g \beta d^2}{\eta}\,\frac{(2\pi k d)^2}{\left[(2\pi k d)^2+\pi^2\right]^2} - \kappa\left[\left(\frac{\pi}{d}\right)^2+(2\pi k)^2\right]$$

$$\alpha' = \frac{\kappa}{d^2}\left[\frac{\rho_0 \alpha g d^4 \beta}{\eta \kappa}\left[\frac{(2\pi k d)^2}{(2\pi k d)^2+\pi^2}\right] - \left[\pi^2+(2\pi k d)^2\right]\right]$$

What parameters make $\alpha' > 0$?

The non-dimensional combination of parameters is called the Rayleigh Number.

$$Ra = \frac{\rho_0 \alpha g d^3 (T_L - T_u)}{\chi \eta}$$

If we re-write the growth rate factor α as

$$\frac{\alpha d^2}{\kappa} = \frac{Ra(2\pi kd)^2 - \left[(2\pi kd)^2 + \pi^2\right]^3}{(2\pi kd)^2 + \pi^2}$$

An instability develops if $\alpha > 0$

$$Ra > \frac{\left[(2\pi kd)^2 + \pi^2\right]^3}{(2\pi kd)^2}$$

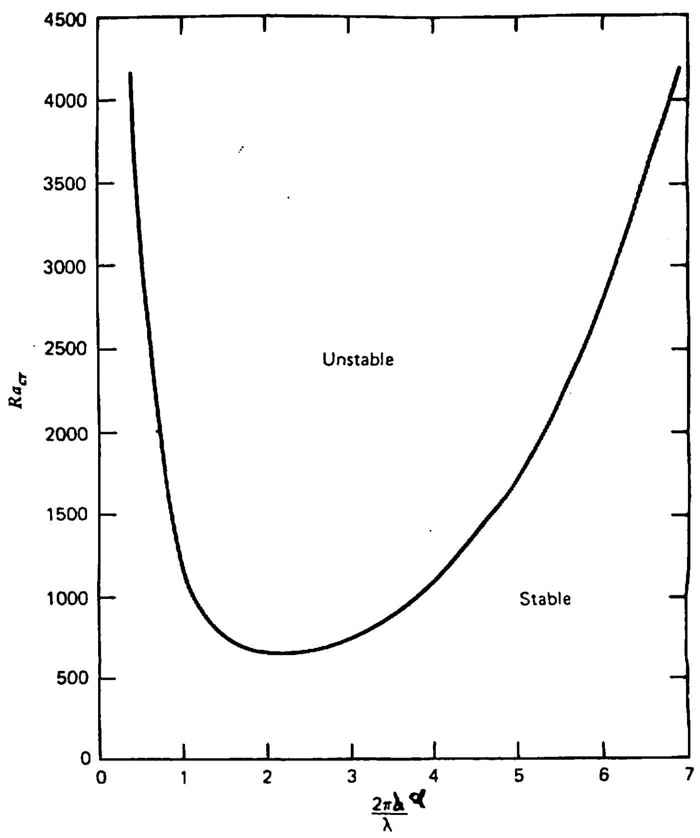
To find minimum of curve take

$$\frac{\partial Ra}{\partial k} = 0$$

$$2\pi kd = \frac{\pi}{\sqrt{2}}$$

$$\lambda = 2\sqrt{2}\, d$$

$$Ra_{cr} = \frac{27\pi^4}{4} = 657.5$$



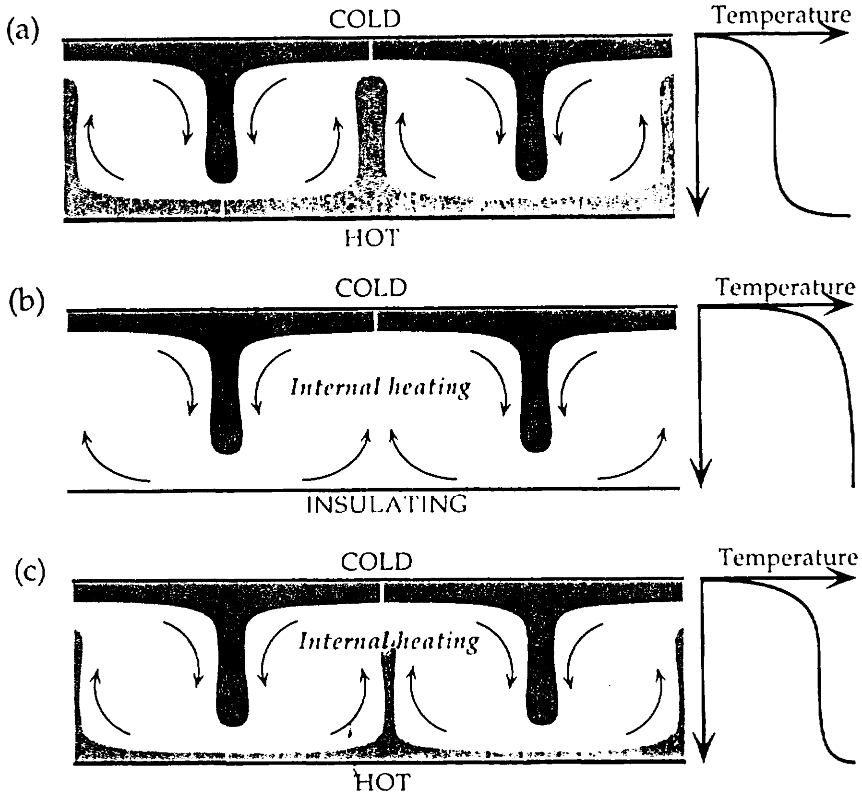


Figure 8.3. Sketches illustrating how the existence and strength of a lower thermal boundary layer depend on the way in which the fluid layer is heated.

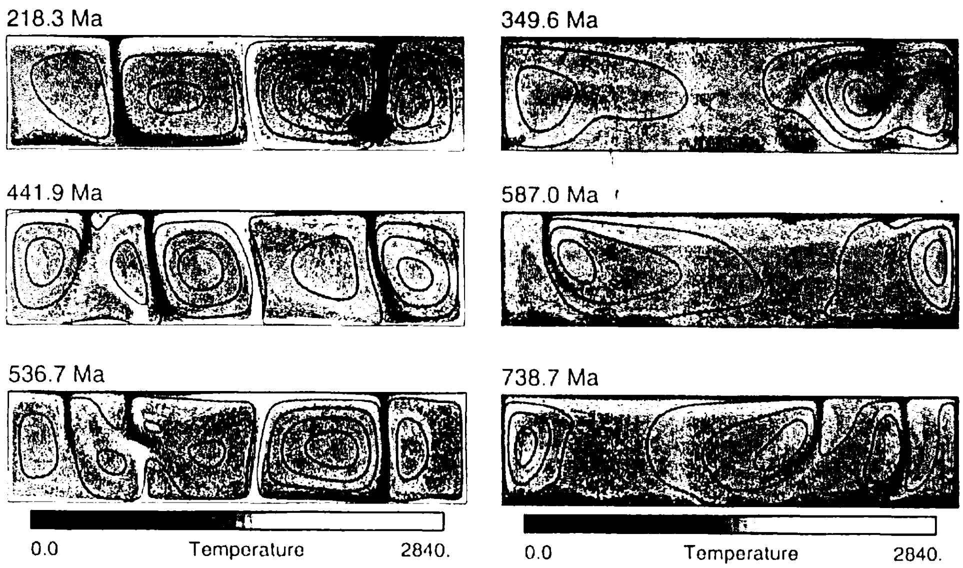


Figure 8.4. Frames from numerical models, illustrating the differences between convection in a layer heated from below (left-hand panels) and in a layer heated internally (right-hand panels). (Technical specifications of these models are given in Appendix 2.)

10.1029/2001GC000238

Table 6. Results of Large-Scale Models (2-D)

Model	V1	V2	V3	K2	K3	K4	K8	H1	P1	P2
V_{OC} (cm/yr)	5	10	2.5	5	5	5	5	5	5	5
A_{OC} (Myr)	130	65	260	130	130	130	130	130	130	130
δ_η	0	0	0	1	2	3	7	0	0	2
H	0	0	0	0	0	0	0	18	0	18
Phase change	No	No	No	No	No	No	No	No	Yes	Yes
W_s (km)	498	459	546	543	543	543	543	467	645	546
T_s	0.30	0.31	0.31	0.31	0.34	0.37	0.40	0.30	0.28	0.34
Temperature anomaly $(T_m - T_s)\Delta T$ (°C)	580	551	551	551	464	377	290	580	638	464

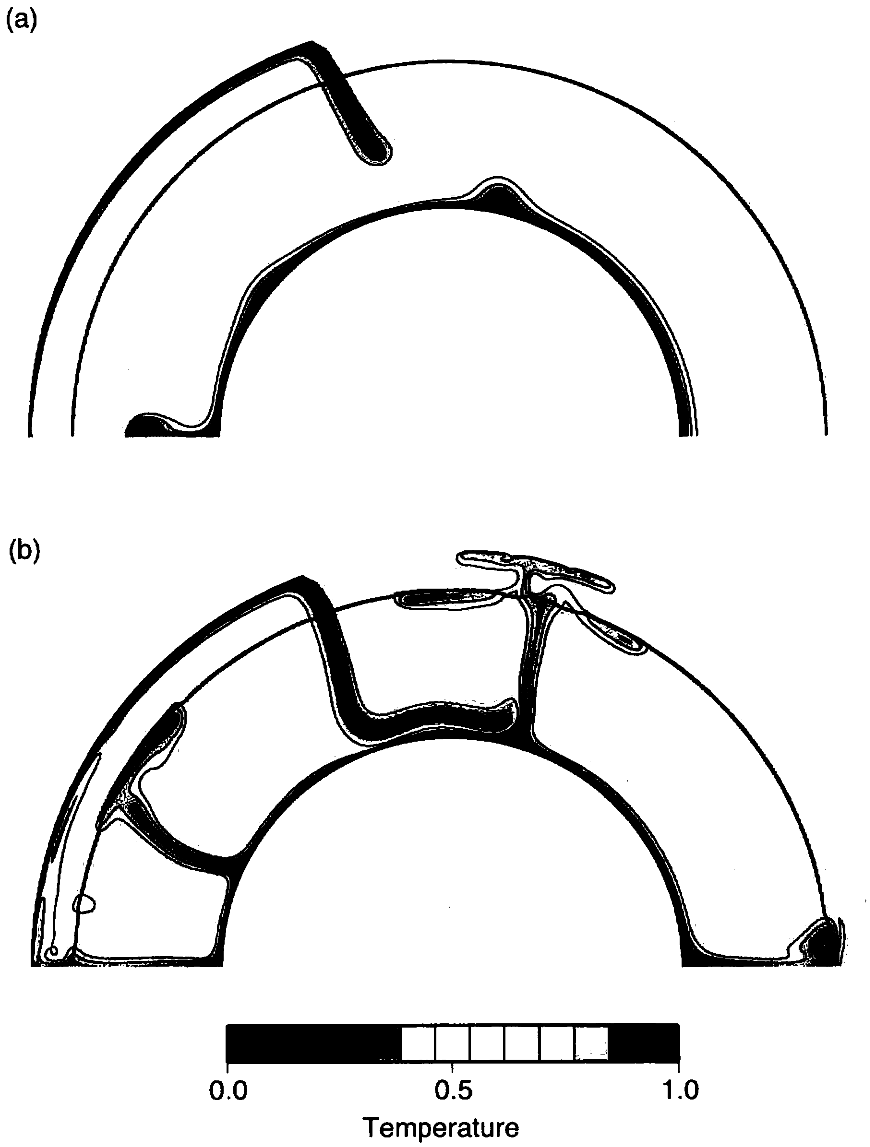


Figure 4. Temperature fields of Model P2. The green lines mark the 660-km phase boundary. (a) When the slab descends through the lower mantle, it induces a down-welling flow. This flow depresses the thickness of the TBL directly beneath the slab and pushes hot materials aside, thus thickening the neighbored TBL, even when the slab is still relatively distant from the CMB. The thickened TBL is prone to instability and initiates the growth of a new plume. (b) As the tip of the slab reaches the CMB, the slab slides horizontally while sweeping hot material aside (including the plume root)