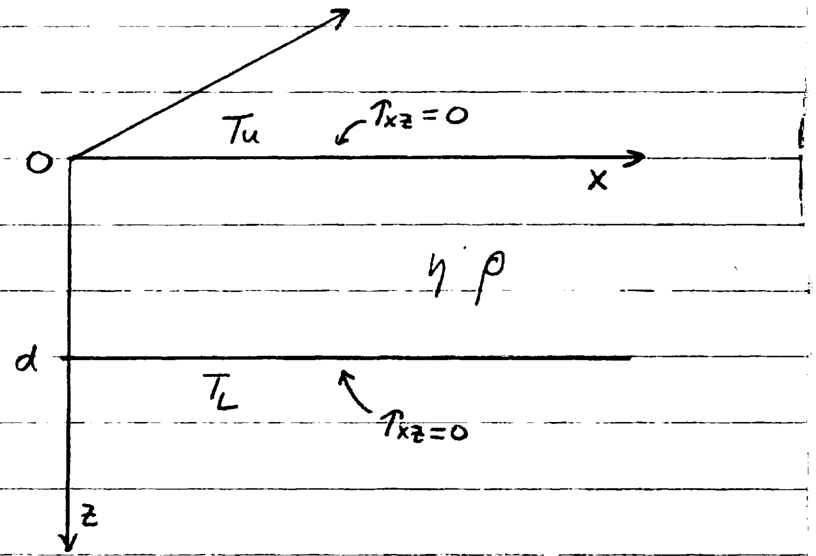
Onset of convection and the Rayleigh Number

(T&S, p 267-272)

Start with a fluid layer of viscosity η and density ρ.
Hold the temperature at the top fixed T_u and increase T_L until convection first develops.



What temperature difference is needed?
What is the wavelength of the convection?

Assume free-slip at top and bottom.
Assume 2-D flow.
Incompressible fluid.

Basic State $\quad u_0 = w_0 = 0 \qquad T_0 = T_u + \beta z \qquad \beta = \dfrac{T_L - T_u}{d}$

$$\frac{\partial P_0}{\partial x} = 0 \qquad \frac{\partial P_0}{\partial z} = g\rho$$

Perturbation $\quad T = T_0 + \theta$

$\rho = \rho_0(1 - \alpha\theta)$ Boussinesq approximation

$P = P_0 + \delta P$

u, w these are small

Bénard-Marangoni Convection in Two Layered Liquids

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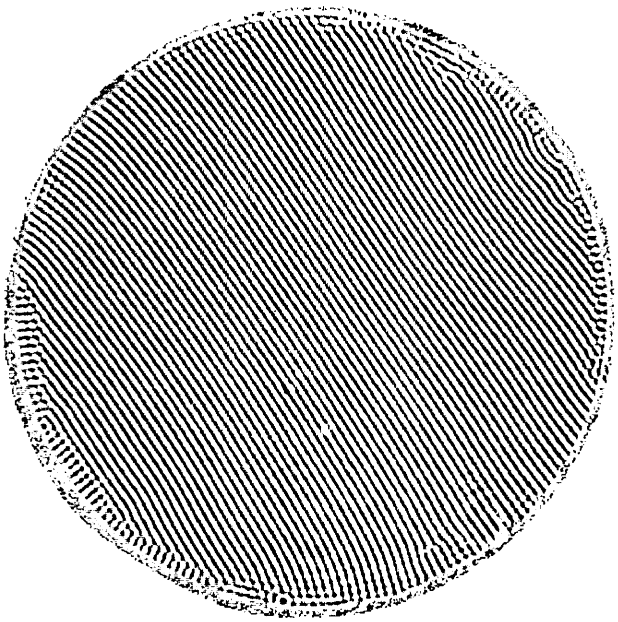
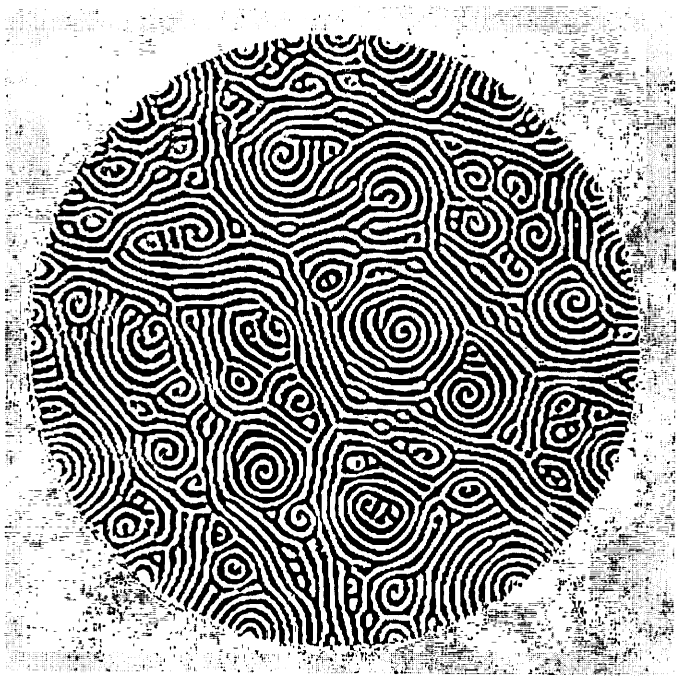
We describe experiments on Bénard-Marangoni convection in horizontal layers of two immiscible liquids. Unlike previous experiments, which used gases as the upper fluid, we find a square planform close to onset which undergoes a secondary bifurcation to rolls at higher temperature differences. The scale of the convection pattern is that of the thinner lower fluid layer for which buoyancy and surface tension forces are comparable. The wavenumber of the pattern near onset agrees with the prediction of the linear stability analysis for the full two-layer problem. The square planform is in qualitative agreement with recent one- and two-layer nonlinear theories, which fail however to predict the transition to rolls.





mass $\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0$

momentum

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho_0} \nabla p + \frac{\eta}{\rho_0} \nabla^2 \vec{u} + \alpha g \theta \hat{k}$$

energy $\dfrac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla \theta = \kappa \nabla^2 \theta$

Terms with products of small quantities can be neglected. Also assume marginal stability so the acceleration terms are small (infinite Pr) but the perturbation will grow or die. Convective rolls develop at marginal instability so use 2-D.

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\eta \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\partial p}{\partial x} = 0$$

$$\eta \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{\partial p}{\partial z} - \alpha g \rho_0 \theta = 0$$

$$\kappa \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2} \right) - w \beta = \frac{\partial \theta}{\partial t}$$

introduce stream function

$$u = -\frac{\partial \psi}{\partial z} \qquad w = \frac{\partial \psi}{\partial x}$$

take x-derivative of z equation and z-derivative of x-equation and add

$$\eta\left(\frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial z^2} + \frac{\partial^4 \psi}{\partial z^4}\right) - \rho_0 \alpha g \frac{\partial \theta}{\partial x} = 0$$

$$\kappa\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2}\right) = \beta \frac{\partial \psi}{\partial x} + \frac{\partial \theta}{\partial t}$$

B.C. $\theta(0) = \theta(d) = 0$ $\qquad \frac{\partial \psi}{\partial x} = 0 \quad z = 0, d$ (zero vertical velocity)

$\frac{\partial^2 \psi}{\partial z^2} = 0 \quad z = 0, d$ (zero shear stress)

Assume solutions of the form

$$\psi = \psi_0 \sin\left(\frac{\pi z}{d}\right) \sin(2\pi k x)\, e^{\alpha' t} \qquad \frac{\partial^2 \psi}{\partial z^2} = -\left(\frac{\pi}{d}\right)^2 \psi \qquad \frac{\partial^2 \psi}{\partial x^2} = -(2\pi k)^2 \psi$$

$$\theta = \theta_0 \sin\left(\frac{\pi z}{d}\right) \cos(2\pi k x)\, e^{\alpha' t} \qquad \frac{\partial^2 \theta}{\partial z^2} = -\left(\frac{\pi}{d}\right)^2 \theta \qquad \frac{\partial^2 \theta}{\partial z^2} = -(2\pi k)^2 \theta$$

$\alpha' > 0$ unstable $\qquad \alpha' < 0$ stable

$$\eta\left[(2\pi k)^2+\left(\frac{\pi}{d}\right)^2\right]^2\psi_0 = -\rho_0\alpha g(2\pi k)\,\theta_0 \qquad \text{(momentum)}$$

$$\left[\alpha' + \chi\left(\frac{\pi}{d}\right)^2 + \chi(2\pi k)^2\right]\theta_0 = -2\pi k\,\beta\,\psi_0 \qquad \text{(energy)}$$

from the first equation

$$\psi_0 = \frac{-\rho_0\alpha g(2\pi k)}{\eta}\,\theta_0\left[(2\pi k)^2+\left(\frac{\pi}{d}\right)^2\right]^{-2}$$

plus into second equation

$$\left[\alpha' + \chi\left(\frac{\pi}{d}\right)^2 + \chi(2\pi k)^2\right]\phi_0 = \frac{(2\pi k)^2\rho_0\alpha g\beta}{\eta}\,\phi_0\left[(2\pi k)^2+\left(\frac{\pi}{d}\right)^2\right]^{-2}$$

$$\alpha' = \frac{\rho_0\alpha g\beta d^2}{\eta}\,\frac{(2\pi kd)^2}{\left[(2\pi kd)^2+\pi^2\right]^2} - \chi\left[\left(\frac{\pi}{d}\right)^2+(2\pi k)^2\right]$$

$$\alpha' = \frac{\chi}{d^2}\left[\frac{\rho_0\alpha g d^4\beta}{\eta\chi}\left[\frac{(2\pi kd)^2}{(2\pi kd)^2+\pi^2}\right] - \left[\pi^2+(2\pi kd)^2\right]\right]$$

What parameters make $\alpha' > 0$?

The non-dimensional combination of parameters is called the Rayleigh Number.

$$Ra = \frac{\rho_0 \alpha g d^3 (T_L - T_u)}{\kappa \eta}$$

If we re-write the growth rate factor α as

$$\frac{\alpha d^2}{\kappa} = \frac{Ra\,(2\pi k d)^2 - \left[(2\pi k d)^2 + \pi^2\right]^3}{(2\pi k d)^2 + \pi^2}$$

An instability develops if $\alpha > 0$

$$Ra > \frac{\left[(2\pi k d)^2 + \pi^2\right]^3}{(2\pi k d)^2}$$

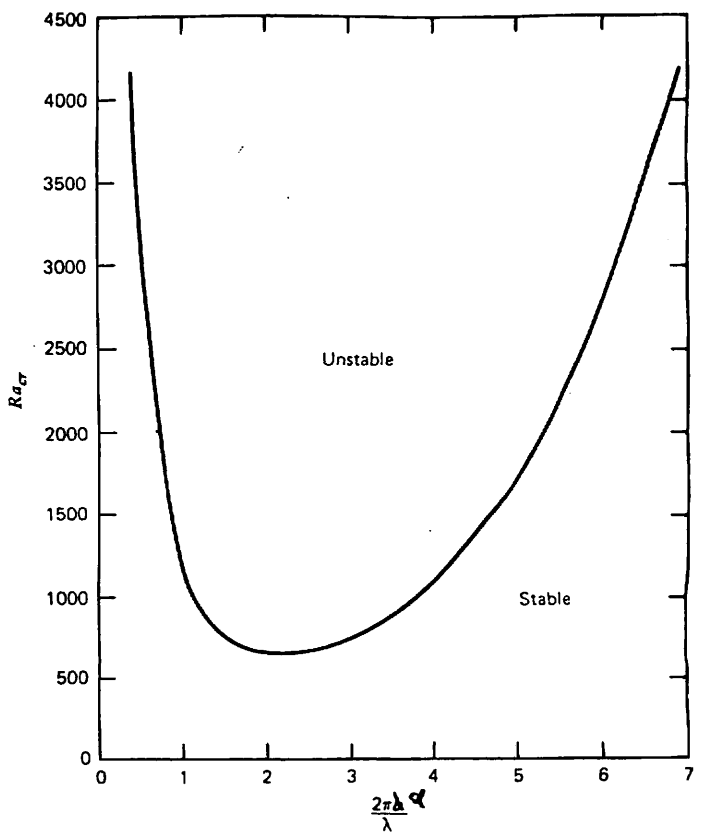
To find minimum of curve take

$$\frac{\partial Ra}{\partial k} = 0$$

$$2\pi k d = \frac{\pi}{\sqrt{2}}$$

$$\lambda = 2\sqrt{2}\, d$$

$$Ra_{cr} = \frac{27\pi^4}{4} = 657.5$$



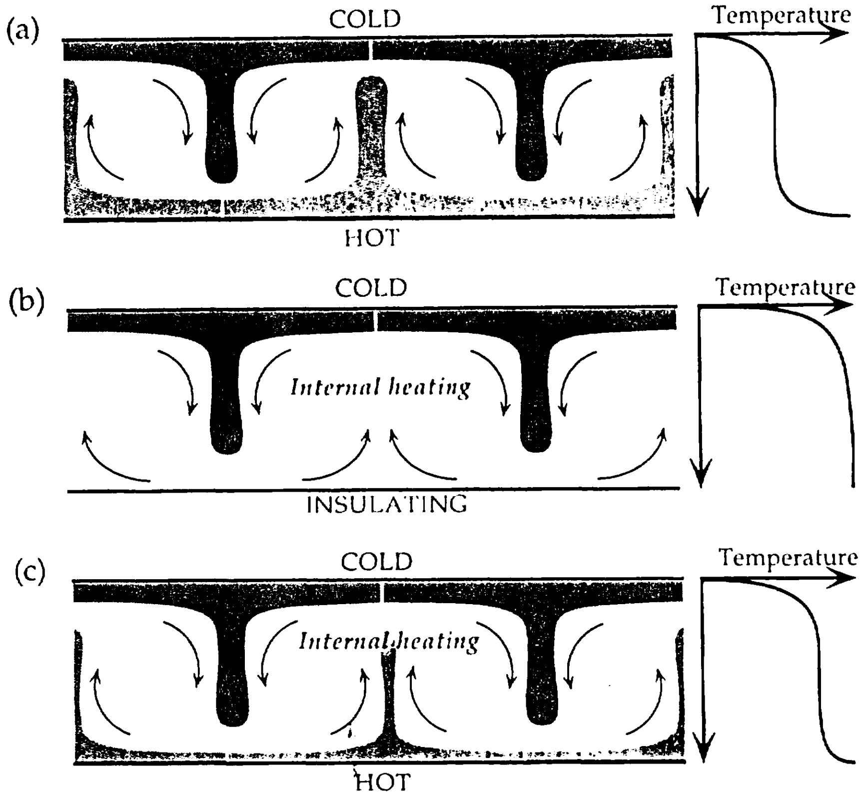


Figure 8.3. Sketches illustrating how the existence and strength of a lower thermal boundary layer depend on the way in which the fluid layer is heated.

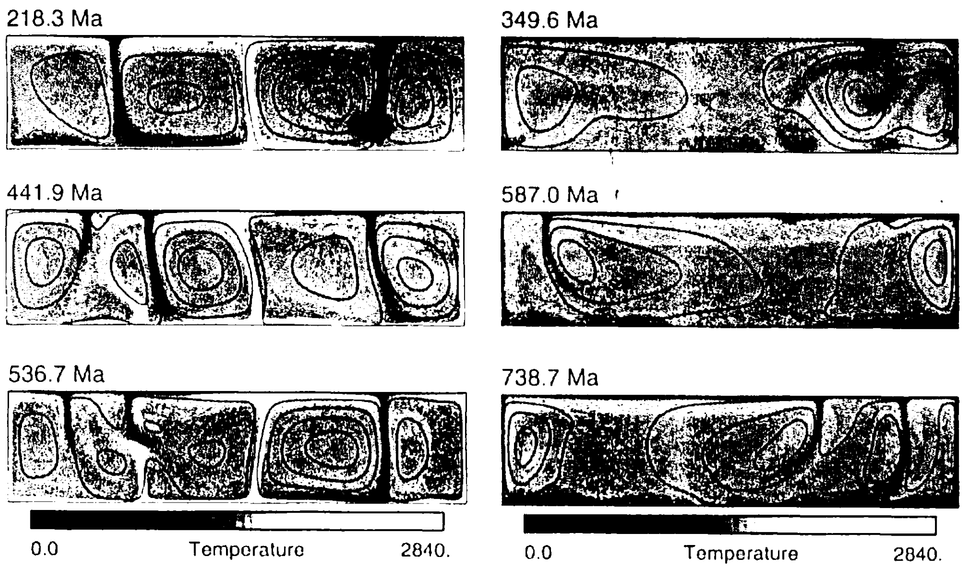


Figure 8.4. Frames from numerical models, illustrating the differences between convection in a layer heated from below (left-hand panels) and in a layer heated internally (right-hand panels). (Technical specifications of these models are given in Appendix 2.)

Table 6. Results of Large-Scale Models (2-D)

Model	V1	V2	V3	K2	K3	K4	K8	H1	P1	P2
V_{OC} (cm/yr)	5	10	2.5	5	5	5	5	5	5	5
A_{OC} (Myr)	130	65	260	130	130	130	130	130	130	130
δ_E	0	0	0	1	2	3	7	0	0	2
H	0	0	0	0	0	0	0	18	0	18
Phase change	No	No	No	No	No	No	No	No	Yes	Yes
W_s (km)	498	459	546	543	543	543	543	467	645	546
T_s	0.30	0.31	0.31	0.31	0.34	0.37	0.40	0.30	0.28	0.34
Temperature anomaly $(T_m - T_s)\Delta T$ (°C)	580	551	551	551	464	377	290	580	638	464

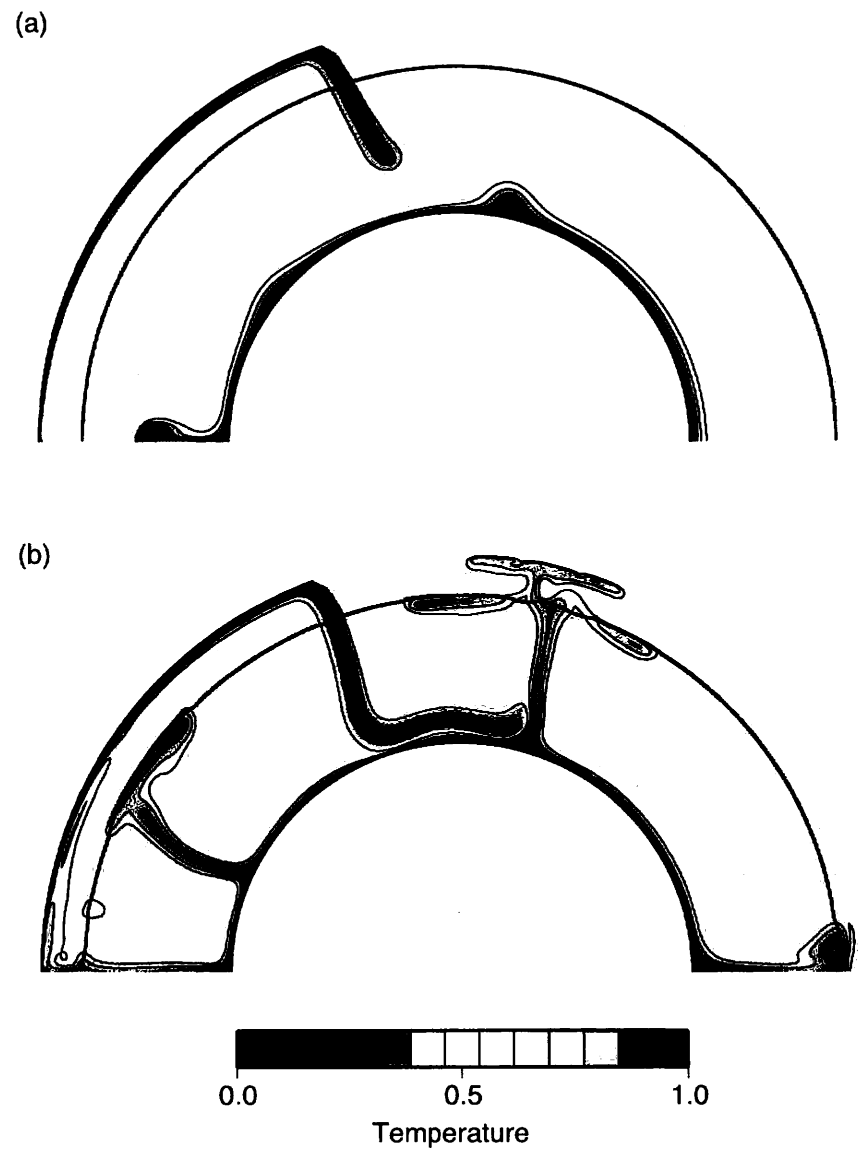


Figure 4. Temperature fields of Model P2. The green lines mark the 660-km phase boundary. (a) When the slab descends through the lower mantle, it induces a down-welling flow. This flow depresses the thickness of the TBL directly beneath the slab and pushes hot materials aside, thus thickening the neighbored TBL, even when the slab is still relatively distant from the CMB. The thickened TBL is prone to instability and initiates the growth of a new plume. (b) As the tip of the slab reaches the CMB, the slab slides horizontally while sweeping hot material aside (including the plume root)