APPLICATIONS AND REVIEW OF FOURIER TRANSFORM/SERIES

(Reference – *The Fourier Transform and its Application, second edition*, R.N. Bracewell, McGraw-Hill Book Co., New York, 1978.)

Fourier analysis is a fundamental tool used in all areas of science and engineering. The fast fourier transform (FFT) algorithm is remarkably efficient for solving large problems. Nearly every computing platform has a library of highly-optimized FFT routines. In the field of Earth science, fourier analysis is used in the following areas:

Solving linear partial differential equations (PDE's):

Gravity/magnetics	Laplace	$\nabla^2\Phi = 0$
Elasticity (flexure)	Biharmonic	$\nabla^4\Phi = 0$
Heat Conduction	Diffusion	$\nabla^2\Phi - \delta\ \Phi/\ \delta t = 0$
Wave Propagation	Wave	$\nabla^2\Phi - \delta^2\Phi/\ \delta t^2 = 0$

Designing and using antennas:
- Seismic arrays and streamers
- Multibeam echo sounder and side scan sonar
- Interferometers – VLBI – GPS
- Synthetic Aperture Radar (SAR) and Interferometric SAR (InSAR)

Image Processing and filters:
- Transformation, representation, and encoding
- Smoothing and sharpening
- Restoration, blur removal, and Wiener filter

Data Processing and Analysis:
- High-pass, low-pass, and band-pass filters
- Cross correlation – transfer functions – coherence
- Signal and noise estimation – encoding time series

In this remote sensing course we will use fourier analysis to understand and evaluate apertures (antennas and telescopes) as well as to filter images.

Fourier analysis deals with complex numbers so perhaps it is time to dust off your book on advanced calculus. Here is a very brief review of the things you'll need. A complex number $z = x + iy$ is composed of real and imaginary numbers. Remember $i = \sqrt{-1}$. Functions can have real and imaginary components as well. For example a general, complex-valued function of a real variable is $f(x) = u(x) + iw(x)$. The most important complex-valued function for this discussion is the complex exponential function $e^{i\theta} = \cos\theta + i\sin\theta$. After a little algebra it is easy to show that $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$. This is the level of math you'll need to understand fourier analysis.

Definitions of fourier transforms in 1-D and 2-D

The 1-dimensional fourier transform is defined as:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi kx} dx \qquad F(k) = \Im[f(x)] \quad \text{- forward transform}$$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{i2\pi kx} dk \qquad f(x) = \Im^{-1}[F(k)] \quad \text{- inverse transform}$$

where x is distance and k is wavenumber where $k = 1/\lambda$ and λ is wavelength. These equations are more commonly written in terms of time t and frequency ν where $\nu = 1/T$ and T is the period.

The 2-dimensional fourier transform is defined as:

$$F(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x}) e^{-i2\pi(\mathbf{k}\cdot\mathbf{x})} d^2\mathbf{x} \qquad F(\mathbf{k}) = \Im_2[f(\mathbf{x})]$$

$$f(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mathbf{k}) e^{i2\pi(\mathbf{k}\cdot\mathbf{x})} d^2\mathbf{k} \qquad f(\mathbf{x}) = \Im_2^{-1}[F(\mathbf{k})]$$

where $\mathbf{x} = (x, y)$ is the position vector, $\mathbf{k} = (k_x, k_y)$ is the wavenumber vector, and $(\mathbf{k} \bullet \mathbf{x}) = k_x x + k_y y$.

The next two page show some examples of fourier transform pairs. These figures were taken from *Bracewell* [1978].

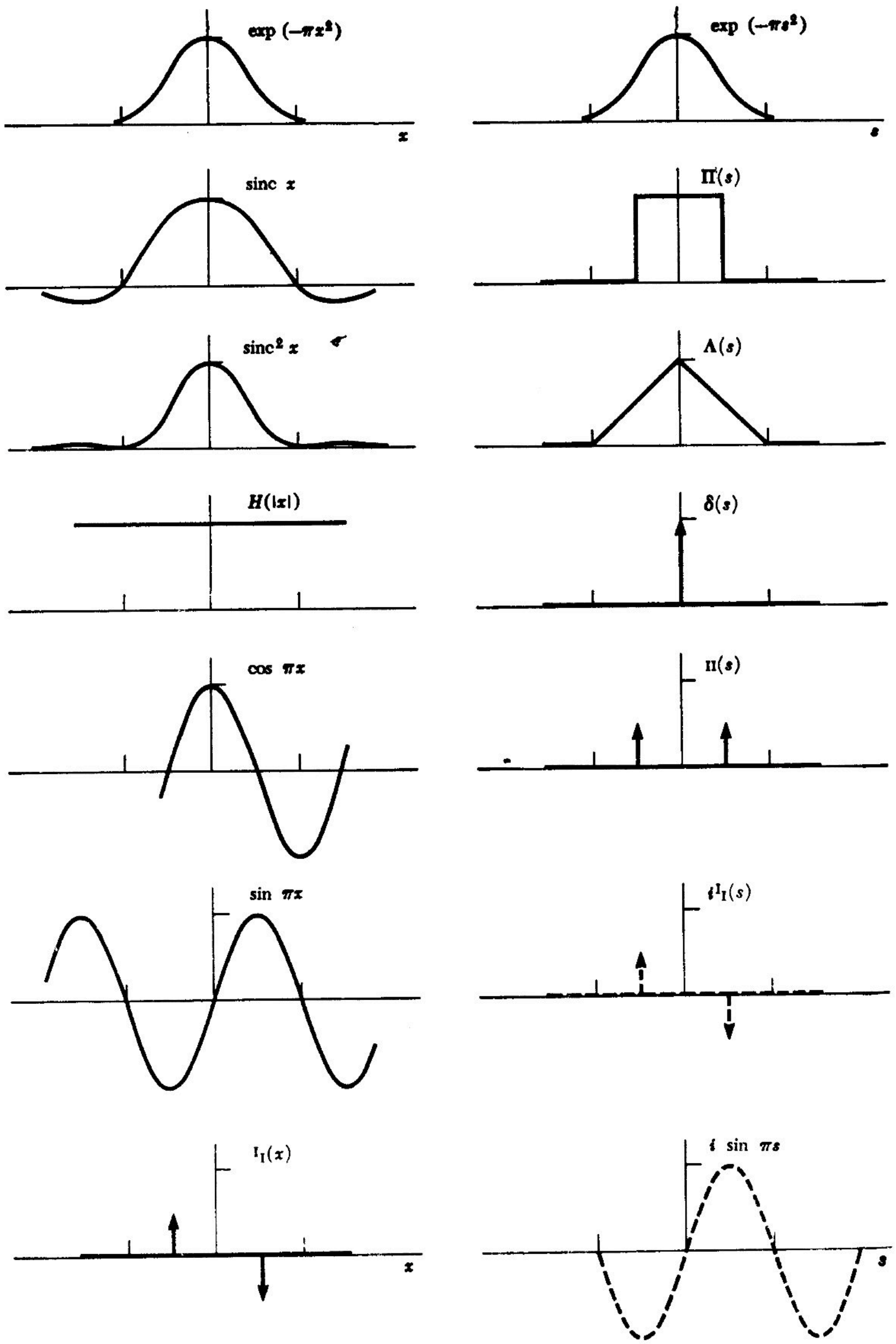
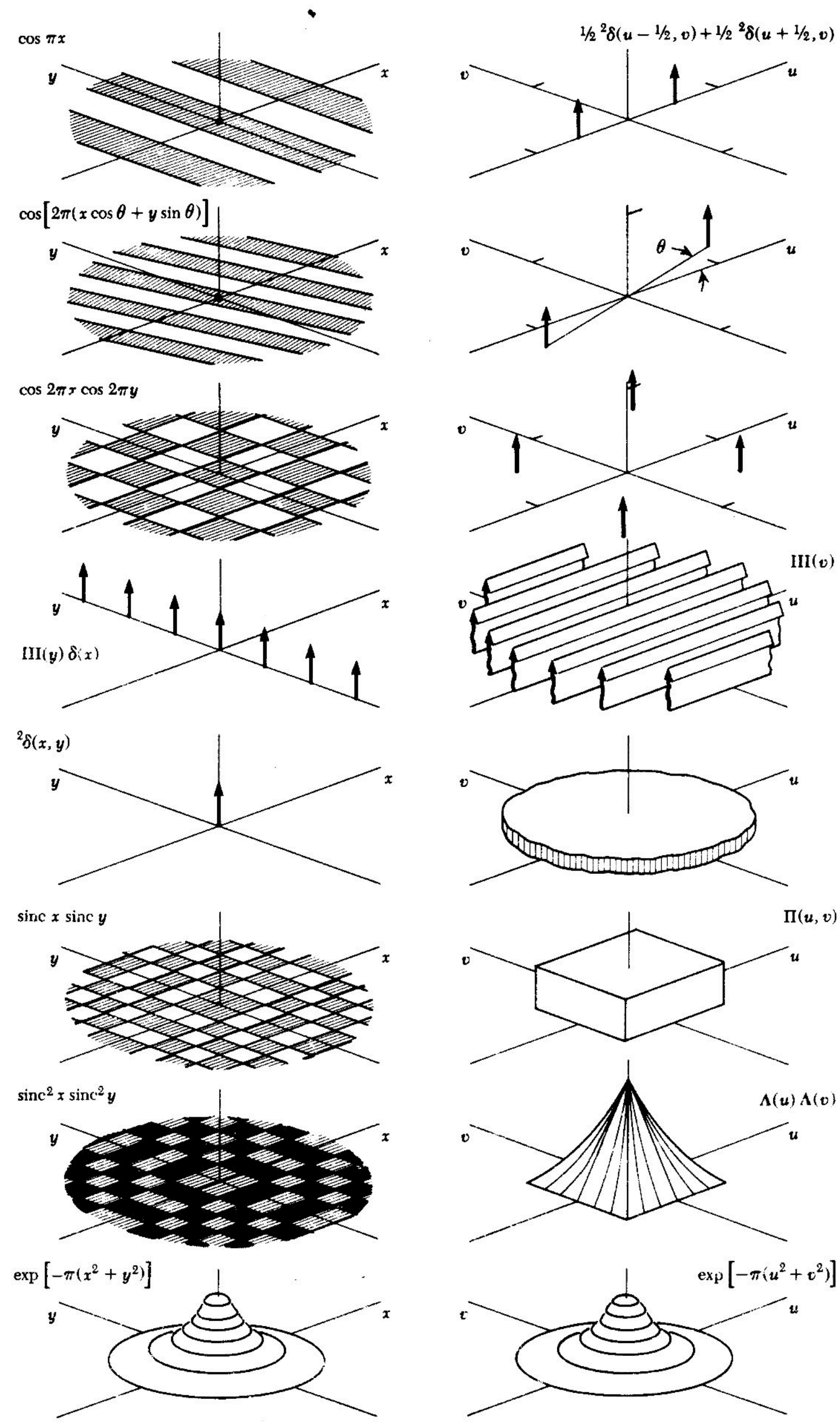


Fig. 6.1 Some Fourier transform pairs for reference.


$\cos \pi x$
$\frac{1}{2}\,{}^2\delta(u - \frac{1}{2}, v) + \frac{1}{2}\,{}^2\delta(u + \frac{1}{2}, v)$
$\cos\left[2\pi(x\cos\theta + y\sin\theta)\right]$
θ
$\cos 2\pi x \cos 2\pi y$
$\mathrm{III}(y)\,\delta(x)$
$\mathrm{III}(v)$
${}^2\delta(x, y)$
$\mathrm{sinc}\, x\, \mathrm{sinc}\, y$
$\Pi(u, v)$
$\mathrm{sinc}^2 x\, \mathrm{sinc}^2 y$
$\Lambda(u)\,\Lambda(v)$
$\exp\left[-\pi(x^2 + y^2)\right]$
$\exp\left[-\pi(u^2 + v^2)\right]$
x
y
u
v

Fourier sine and cosine transforms

Any function *f(x)* can be decomposed into odd *O(x)* and even *E(x)* components.

$$f(x) = E(x) + O(x)$$

$$E(x) = \frac{1}{2}\left[f(x) + f(-x)\right] \qquad O(x) = \frac{1}{2}\left[f(x) - f(-x)\right]$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi kx} dx$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$F(k) = \underbrace{\int_{-\infty}^{\infty} f(x)\cos(2\pi kx)dx}_{\text{odd part cancels}} - i\underbrace{\int_{-\infty}^{\infty} f(x)\sin(2\pi kx)dx}_{\text{even part cancels}}$$

$$F(k) = 2\underbrace{\int_{o}^{\infty} E(x)\cos(2\pi kx)dx}_{\text{cosine transform}} - 2i\underbrace{\int_{o}^{\infty} O(x)\sin(2\pi kx)dx}_{\text{sine transform}}$$

You have probably seen fourier cosine and sine transforms, but it is better to use the complex exponential form.

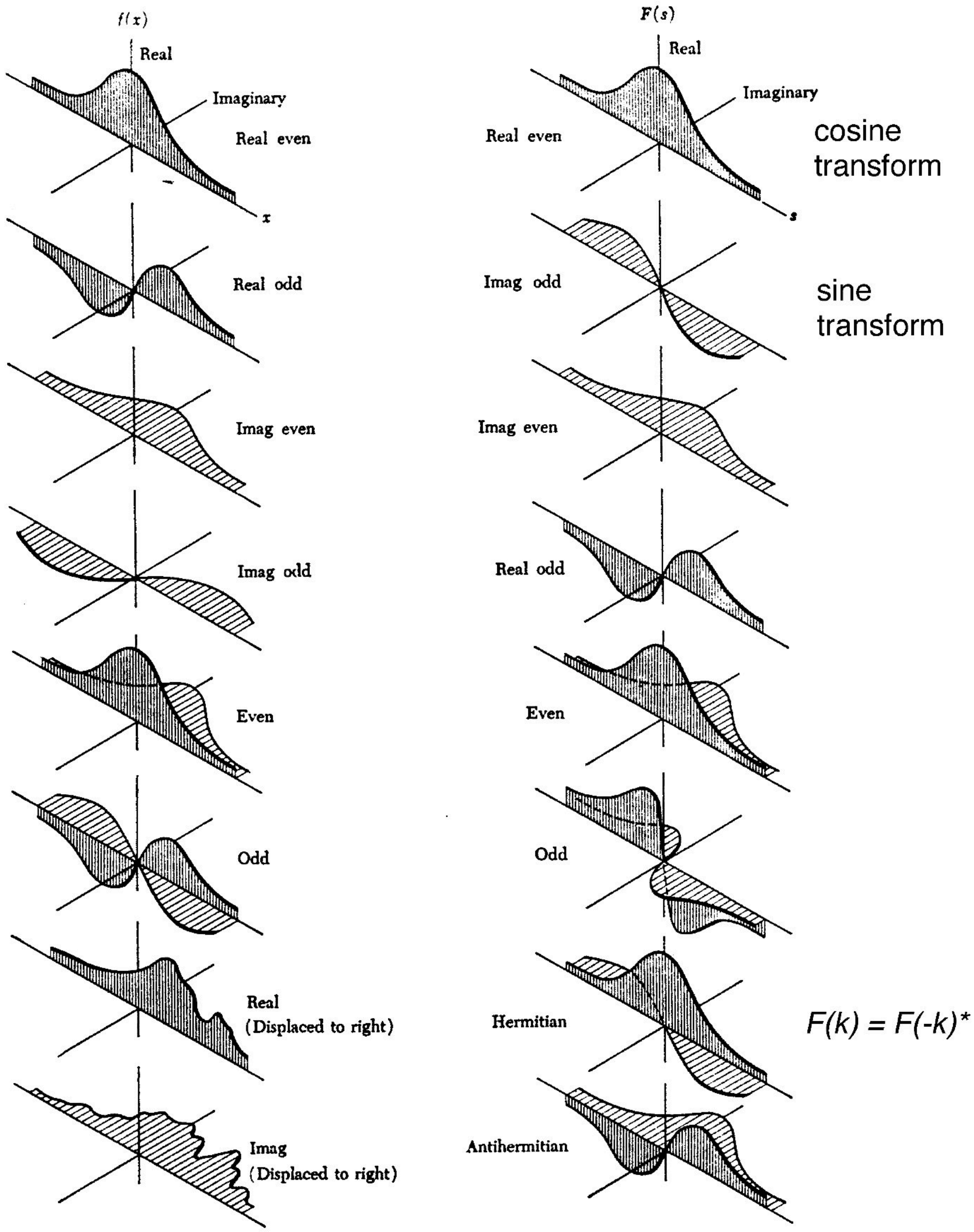


***Fig. 2.5** Symmetry properties of a function and its Fourier transform.*

(scan 600DPI line art)

Properties of fourier transforms

The following are some important properties of fourier transforms that you should derive for yourself at least once. You'll find derivations in Bracewell. Once you have derived and understand these properties, you can treat them as tools. Very complicated problems can be simplified using these tools. For example, when solving a linear partial differential equation, one uses the derivative property to reduce the differential equation to an algebraic equation.

similarity property $\Im\left[f(ax)\right] = \frac{1}{|a|}F\left(\frac{k}{a}\right)$

shift property $\Im\left[f(x-a)\right] = e^{-i2\pi ka}F(k)$

differentiation property $\Im\left[\frac{df}{dx}\right] = i2\pi k\,F(k)$

convolution property $\Im\left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right] = F(k)G(k)$

Rayleigh's theorem $\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(k)|^2\,dk$

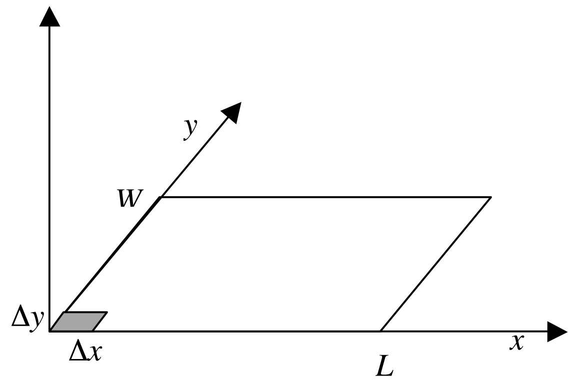
Note: These properties are equally valid in 2-dimensions or even n-dimensions. The properties also apply to discrete data. See Chapter 18 in Bracewell.

Fourier series

Many geophysical problems are concerned with a small area on the surface of the Earth.

W - width of area

L - length of area



The coefficients of the 2-dimensional Fourier series are computed by the following integration.

$$F_n^m = \frac{1}{LW}\int_o^L\int_o^W f(x,y)\exp\left[-i2\pi\left(\frac{m}{L}x + \frac{n}{W}y\right)\right]dydx$$

The function is reconstructed by the following summations over the fourier coefficients.

$$f(x,y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} F_n^m \exp\left[i2\pi\left(\frac{m}{L}x + \frac{n}{W}y\right)\right]$$

The finite size of the area leads to a discrete set of wavenumbers $k_x = m/L$, $k_y = n/W$ and a discrete set of fourier coefficients F_n^m. In addition to the finite size of the area, geophysical data commonly have a characteristic sampling interval Δx and Δy.

$I = L/\Delta x$ - number of points in the x-direction

$J = W/\Delta y$ - number of points in the y-direction

The Nyquist wavenumbers is $k_x = 1/(2\,\Delta x)$ and $k_y = 1/(2\,\Delta y)$ so there is a finite set of fourier coefficients $-I/2 < m < I/2$ and $-J/2 < n < J/2$. Recall the trapesoidal rule of integration.

$$\int_o^L f(x)dx \cong \sum_{i=0}^{I-1} f(x_i)\Delta x \quad \text{where } x_i = i\Delta x.$$

$$\int_o^L f(x)dx \cong \frac{L}{I}\sum_{i=0}^{I-1} f(x_i)$$

The discrete forward and inverse fourier transform are:

$$F_n^m = \frac{1}{IJ}\sum_{i=o}^{I-1}\sum_{j=o}^{J-1} f_i^j \exp\left[-i2\pi\left(\frac{m}{I}i + \frac{n}{J}j\right)\right]$$

$$f_i^j = \sum_{n=-I/2}^{I/2}\ \sum_{m=-J/2}^{J/2} F_n^m \exp\left[i2\pi\left(\frac{i}{I}m + \frac{j}{J}n\right)\right]$$

The forward and inverse discrete fourier transforms are almost identical sums so one can use the same computer code for both operations.