FOURIER TRANSFORM METHODS

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1. Fourier Transforms

Fourier analysis is a fundamental tool used in all areas of science and engineering. The fast fourier transform (FFT) algorithm is remarkably efficient for solving large problems. Nearly every computing platform has a library of highly-optimized FFT routines. In the field of Earth science, fourier analysis is used in the following areas:

Solving linear partial differential equations (PDE's):

Gravity/magnetics	Laplace	$\nabla^2\Phi = 0$
Elasticity (flexure)	Biharmonic	$\nabla^4\Phi = 0$
Heat Conduction	Diffusion	$\nabla^2\Phi - \delta\Phi/\delta t = 0$
Wave Propagation	Wave	$\nabla^2\Phi - \delta^2\Phi/\delta t^2 = 0$

Designing and using antennas:

Seismic arrays and streamers

Multibeam echo sounder and side scan sonar

Interferometers – VLBI – GPS

Synthetic Aperture Radar (SAR) and Interferometric SAR (InSAR)

Image Processing and filters:

Transformation, representation, and encoding

Smoothing and sharpening

Restoration, blur removal, and Wiener filter

Data Processing and Analysis:

High-pass, low-pass, and band-pass filters

Cross correlation – transfer functions – coherence

Signal and noise estimation – encoding time series

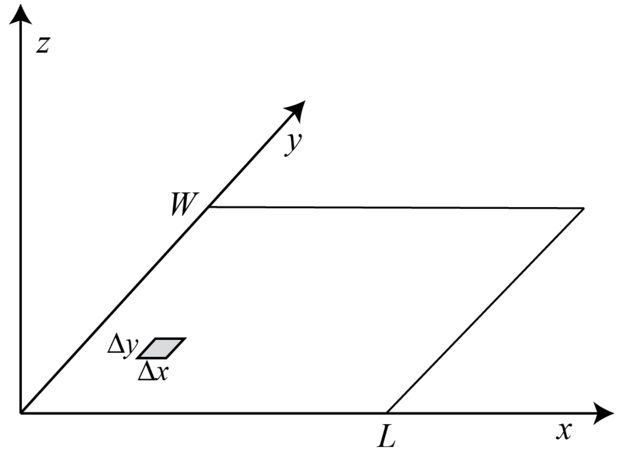

Figure 1.1 Cartesian coordinate system used throughout the book with z positive up. The $z = 0$ plane is the surface of the earth. Fourier transforms deal with infinite domains while the fourier series (section 1.6) has finite domains. For our numerical examples we will select an area of length L and with W consisting of uniform cells of size Δx and Δy. This can be represented as a 2-D array of numbers with $J = L / \Delta x$ columns and $I = W / \Delta y$ rows.

1.1 Definitions of fourier transforms

The 1-dimensional forward and inverse fourier transforms are defined as:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi kx} dx \quad \text{or} \quad F(k) = \Im\left[f(x)\right] \tag{1.1}$$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{i2\pi kx} dk \quad \text{or} \quad f(x) = \Im^{-1}\left[F(k)\right] \tag{1.2}$$

where x is distance and k is wavenumber where $k = 1/\lambda$ and λ is wavelength. We also use the shorthand notation introduced by *Bracewell* [1978]. These equations are more commonly written in terms of time t and frequency ν where $\nu = 1/T$ and T is the period. The 2-dimensional forward and inverse fourier transforms are defined as:

$$F(\mathbf{k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x}) e^{-i2\pi \mathbf{k}\cdot\mathbf{x}} d^2\mathbf{x} \quad \text{or} \quad F(\mathbf{k}) = \Im_2\left[f(\mathbf{x})\right] \tag{1.3}$$

$$f(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\mathbf{k}) e^{i2\pi\mathbf{k}\bullet\mathbf{x}} d^2\mathbf{k} \quad \text{or} \quad f(\mathbf{x}) = \Im_2^{-1}\left[F(\mathbf{k})\right] \tag{1.4}$$

where $\mathrm{x} = (x, y)$ is the position vector, $\mathrm{k} = (k_x, k_y)$ is the wavenumber vector, and $\mathrm{k} \bullet \mathrm{x} = k_x x + k_y y$. The magnitude of the spatial wavenumber $\beta = \left(k_x^2 + k_y^2\right)^{1/2}$ is used often in later chapters. For several of the derivations, we'll also take the fourier transform in the *z*-direction (i.e. 3-D transform) using the following notation.

$$F(k_z) = \int_{-\infty}^{\infty} f(z) e^{-i2\pi k_z z} dz \tag{1.5}$$

$$f(z) = \int_{-\infty}^{\infty} F(k_z) e^{i2\pi k_z z} dk_z \tag{1.6}$$

Fourier transformation with respect to time is also sometimes used to form a 4-D transform.

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\nu t} dt \tag{1.7}$$

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{i2\pi\nu t} d\nu \tag{1.8}$$

While algebraic manipulation of equations in 4-D is sometimes challenging and error prone, we'll use computers to help us in two ways. We'll use the tools of computer algebra to solve the most challenging algebraic manipulations associated with the 3-D and 4-D problems in chapter 4. In addition we'll provide a FORTRAN subroutine called fourt() which can perform numerical fourier transforms in *N*-dimensions although we'll use only 1-, 2-, and 3-D numerical transforms in this book.

1.2 Fourier sine and cosine transforms

Here we introduce the sine and cosine transforms to illustrate the transforms of odd and even functions. Also in later chapters we'll use sine and cosine transforms to match asymmetric and symmetric boundary conditions for particular models.

Any function $f(x)$ can be decomposed into odd $O(x)$ and even $E(x)$ functions such that

$$f(x) = E(x) + O(x) \tag{1.9}$$

where $E(x) = \frac{1}{2}\left[f(x) + f(-x)\right]$ and $O(x) = \frac{1}{2}\left[f(x) - f(-x)\right]$. Note the complex exponential function can be written as

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{1.10}$$

Exercise 1.1 Use equation (1.10) to show that $\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$.

Using this expression (1.10) we can write the forward 1-D transform as the sum of two parts

$$F(k) = \int_{-\infty}^{\infty} f(x)\cos(2\pi kx)\,dx - i\int_{-\infty}^{\infty} f(x)\sin(2\pi kx)\,dx \tag{1.11}$$

After inserting equation (1.9) into this expression and noting that the integral of an odd function times an even function is zero, we arrive at the expressions for the cosine and sine transforms.

$$F(k) = 2\int_{0}^{\infty} E(x)\cos(2\pi kx)\,dx - 2i\int_{0}^{\infty} O(x)\sin(2\pi kx)\,dx \tag{1.12}$$

Throughout this book we'll be dealing with real-valued functions. From equation (1.12) it is evident that the cosine transform of a real, even function is also real and even. Also the sine transform of a real odd function is imaginary and odd. In other words when a function in the space domain is real valued, its fourier transform $F(k)$ has a special Hermitian property $F(k) = F(-k)^*$ so one can reconstruct the transform of the function with negative wavenumbers from the transform with positive wavenumbers. Later when we perform numerical examples using real valued functions such as topography, we can use this Hermitian property to reduce the memory allocation for the fourier transformed array by a factor of 2. This is important for large 2-D and 3-D transforms.

1.3 Examples of Fourier Transforms

Throughout the book we will work with only linear partial differential equations so all the problems are separable and the order of differentiation and integration is irrelevant. For example the 2-D fourier transform of is given by

$$F\left(k_x,k_y\right)=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f\left(x,y\right)e^{-i2\pi k_x x}\,dx\right]e^{-i2\pi k_y y}dy=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f\left(x,y\right)e^{-i2\pi k_y y}\,dy\right]e^{-i2\pi k_x x}dx \qquad (1.13)$$

Note that this 2-D transform consists of a sequence of 1-D transforms. This property can be extended to 3-D, 4-D and even *N*-D; each transform can be performed separately and independently of the transforms in the other dimensions. In the following analysis we'll only show examples of 1-D transforms but the extension to higher dimensions is trivial.

Delta function – By definition the delta function has the following property;

$$\int_{-\infty}^{\infty} f\left(x\right)\delta\left(x-a\right)dx \equiv f\left(a\right) \qquad (1.14)$$

under integration it extracts the value of $f\left(x\right)$ at the position $x=a$. One can describe the delta function as having infinite height at zero argument and zero height elsewhere. The area under the delta function is 1. In terms of pure mathematics the delta function is not a function and only has meaning when integrated against another function. In this book we use the delta function as a powerful tool, provided to us by the mathematicians, so we trust all the mathematical theory behind it. What is the fourier transform of a delta function? By definition if one performs a forward transform of a function followed by an inverse transform, or vice versa, one will arrive back with the original function. Lets try this using the delta function. By definition, the inverse transform of a delta function is

$$\int_{-\infty}^{\infty} \delta(k-k_o) e^{i2\pi kx} dk = e^{i2\pi k_o x} \tag{1.15}$$

Next lets take the forward transform of equation (1.15). The left hand side will be the delta function because we have performed an inverse and forward transform. The right hand side is given by

$$\delta(k-k_o) = \int_{-\infty}^{\infty} e^{i2\pi k_o x} e^{-i2\pi kx} dx = \int_{-\infty}^{\infty} e^{-i2\pi(k-k_o)x}\, dx \tag{1.16}$$

This result shows that the fourier basis functions are orthonormal. If we consider the special case of $k_o = 0$ we see that the fourier transform of a delta function is $\Im[\delta(k)] = 1$. Since fourier transformation is reciprocal in distance *x* and wavenumber *k*, it is also true that $\Im[\delta(x)] = 1$. The delta function and its fourier transform provide an amazingly powerful tool for solving linear PDEs.

Cosine and sine functions – Lets use the delta function tool and the expressions from Exercise 1.1 to calculate the fourier transform of a cosine function having a single wavenumber $\cos(2\pi k_0 x)$.

$$\int_{-\infty}^{\infty} \cos(2\pi k_o x) e^{-i2\pi kx} dx = \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{i2\pi k_o x} + e^{-i2\pi k_o x} +\right) e^{-i2\pi kx} dx = \frac{1}{2}\left[\delta(k-k_o) + \delta(k+k_o)\right] \tag{1.17}$$

So the fourier transform of a cosine function is simply two delta functions located at $\pm k_o$.

Exercise 1.2 Show that the Fourier transform of $\sin(2\pi k_0 x)$ is $\frac{1}{2i}\left[\delta(k-k_o) - \delta(k+k_o)\right]$.

Gaussian function – The Gaussian $e^{-\pi x^2}$ function also plays a fundamental role in solutions to several types of PDEs. Its fourier transform is

$$F(k)=\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i2\pi kx} dx=\int_{-\infty}^{\infty} e^{-\pi\left(x^2+i2kx\right)} dx \tag{1.18}$$

Note that $(x+ik)^2=\left(x^2+i2kx\right)-k^2$. Using this we can rewrite equation (1.18) as

$$F(k)=e^{-\pi k^2}\int_{-\infty}^{\infty} e^{-\pi(x+ik)^2} dx=e^{-\pi k^2}\int_{-\infty}^{\infty} e^{-\pi(x+ik)^2} d(x+ik)=e^{-\pi k^2} \tag{1.19}$$

where we have used the result that the infinite integral of $e^{-\pi x^2}$ is 1. This is a remarkable and powerful result that the fourier transform of a Gaussian is simply a Gaussian.

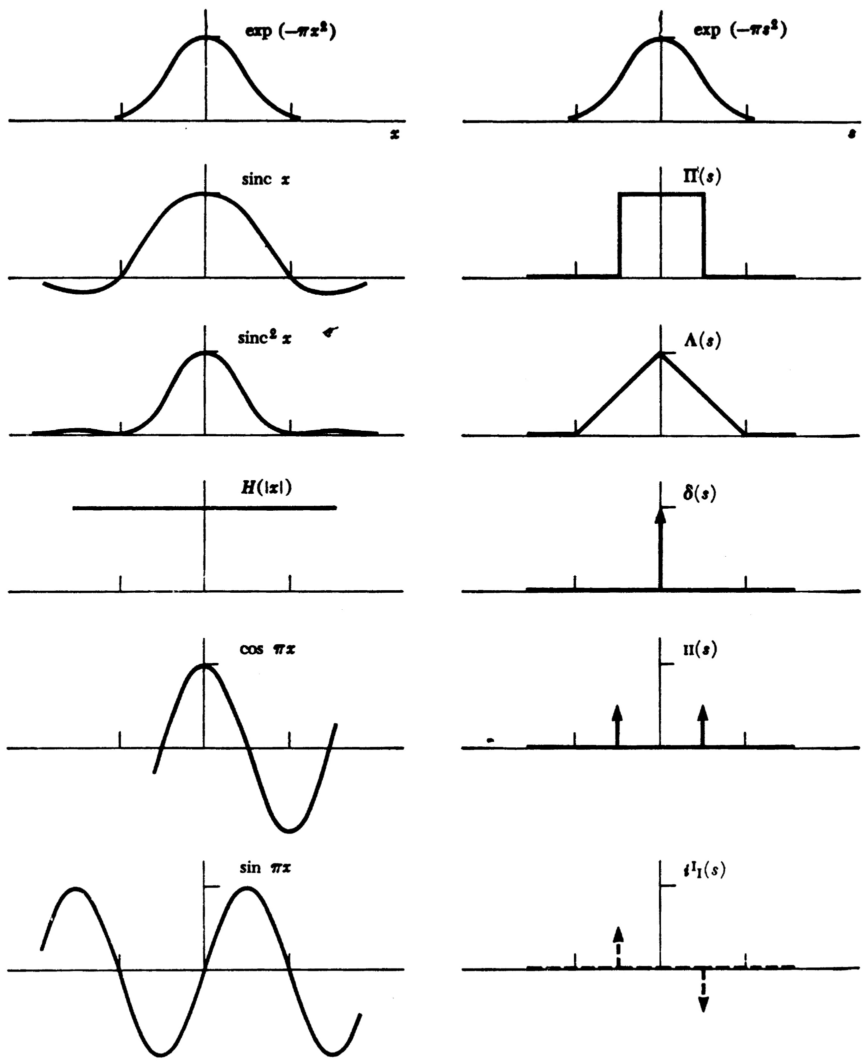

Figure 1.2 Schematic plots of 1-D fourier transform pairs. Solid line is real-valued function while dashed line is imaginary valued function (figure from *Bracewell* [1978].

1.4 Properties of fourier transforms

There are several properties of fourier transforms that can be used as tools for solving PDEs. The first property called the *similarity property* says that if you scale a function by a factor of a along the x-axis, its fourier transform will scaled by a^{-1} along the k-axis and the amplitude will be scaled by $|a|^{-1}$.

Exercise 1.3 Use the definition of the fourier transform equation (1.1) and a change of variable to show the following. Try positive and negative values for *a* to understand why the absolute value is needed in the amplitude scaling.

$$\Im\left[f(ax)\right]=\frac{1}{|a|}F\left(\frac{k}{a}\right) \tag{1.20}$$

The *shift property* says that the fourier transform of a function that is shifted by a along the x-axis equals the original fourier transform scaled by a phase factor. This property is especially useful for numerically shifting a function a non-integer amount of the data spacing along the axis.

Exercise 1.4 Use the definition of the fourier transform and a change of variable to show the following

$$\Im\left[f(x-a)\right]=e^{-i2\pi ka}F(k) \tag{1.21}$$

The *derivative property* of the fourier transform is the essential tool used in this book to transform linear PDEs into algebraic equations that are easily solved. It states that the fourier transform of the derivative of a function is the fourier transform of the original function scaled by the imaginary wavenumber.

$$\Im\left[\frac{\partial f}{\partial x}\right]=i2\pi kF(k) \tag{1.22}$$

To show this is true we start with the inverse transform of equation (1.22)

$$\frac{\partial f}{\partial x}=\int_{-\infty}^{\infty} i2\pi kF(k)e^{i2\pi kx}dk \tag{1.23}$$

Next take the forward transform of equation (1.23) and rearrange terms

$$\Im\left[\frac{\partial f}{\partial x}\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i2\pi k_o F\left(k_o\right)e^{i2\pi k_o x}dk_o e^{-i2\pi kx}dx=\int_{-\infty}^{\infty} i2\pi k_o F\left(k_o\right)\left\{\int_{-\infty}^{\infty} e^{-i2\pi\left(k-k_o\right)x}dx\right\}dk_o \qquad (1.24)$$

The term in the curly brackets is the delta function $\delta\left(k-k_o\right)$ given in equation (1.16). The result is

$$\Im\left[\frac{\partial f}{\partial x}\right]=\int_{-\infty}^{\infty} i2\pi k_o F\left(k_o\right)\delta\left(k-k_o\right)dk_o=i2\pi kF\left(k\right) \qquad (1.25)$$

The final property considered here is the convolution theorem which states that the fourier transform of the convolution of two functions is equal to the product of the fourier transforms of the original functions.

$$\Im\left[\int_{-\infty}^{\infty} f\left(u\right)g\left(x-u\right)du\right]=F\left(k\right)G\left(k\right) \qquad (1.26)$$

To show this is true one can perform the fourier integration on the left side of equation (1.26) and rearrange the order of the integrations

$$\Im\left[\int_{-\infty}^{\infty} f\left(u\right)g\left(x-u\right)du\right]=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f\left(u\right)g\left(x-u\right)du\right]e^{-i2\pi kx}dx=\int_{-\infty}^{\infty} f\left(u\right)\left\{\int_{-\infty}^{\infty} g\left(x-u\right)e^{-i2\pi kx}\,dx\right\}du \qquad (1.27)$$

Next use the shift property of the fourier transform to note that the function in the curly brackets on the right side of equation (1.27) is $e^{-i2\pi ku}G\left(k\right)$. The result becomes

$$\Im\left[\int_{-\infty}^{\infty} f\left(u\right)g\left(x-u\right)du\right]=G\left(k\right)\int_{-\infty}^{\infty} f\left(u\right)e^{-i2\pi ku}du=F\left(k\right)G\left(k\right) \qquad (1.28)$$

Note that these 4 properties are equally valid in 2-dimensions or even N-dimensions. The properties also apply to discrete data. See Chapter 18 in *Bracewell* [1978].